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A dynamic rating curve approach to indirect discharge measurement

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Abstract

The operational measurement of discharge in medium and large rivers is mostly based on indirect approaches by converting water stages into discharge on the basis of steady-flow rating curves. Unfortunately, under unsteady flow conditions, this approach does not guarantee accurate estimation of the discharge due, on the one hand, to the underlying steady state assumptions and, on the other hand, to the required extrapolation of the rating curve beyond the range of actual measurements used for its derivation.

Historically, several formulas were proposed to correct the steady-state discharge value and to approximate the unsteady-flow stage-discharge relationship on the basis of water level measurements taken at a single cross section, where a steady state rating curve is available. However, most of them are either over-simplified or based on approximations that prevented their generalisation. Moreover all the mentioned formulas have been rarely tested on cases where their use becomes essential, namely under unsteady-flow conditions characterised by wide loop rating curves.

In the present work, a new approach, based on simultaneous stage measurements at two adjacent cross sections, is introduced and compared to the approaches described in the literature. The comparison has been carried out on channels with constant or spatially variable geometry under a wide range of flood wave and river bed slope conditions. The results clearly show the improvement in the discharge estimation and the reduction of estimation errors obtainable using the proposed approach.

1 Introduction

Discharge measurement is an issue of major importance for the evaluation of water balance at catchment scale, for the design of water-control and conveyance structures, for rainfall-runoff and flood routing models calibration and validation.

Although several direct measurement approaches exist, only indirect approaches tend to be operationally used in medium and large rivers. Usually, discharge estimates are based on a one-to-one stage-discharge relationship, or steady-flow rating curve, which is derived on the basis of a number of simultaneous stage and discharge measurements. A measure of stage is then directly converted into discharge by means of the developed rating curve.

Such approach can be considered adequate for all rivers under steady-flow conditions as well as, under unsteady-flow conditions, when flood waves show a marked kinematic behaviour, which generally corresponds to rivers with steep bed slopes ($>10^{-3}$). In all the other cases the variable energy slope, associated with the dynamic inertia and pressure forces relevant to the unsteady flow discharge, lead to the formation of an hysteretic rating curve also known as the loop-rating curve (Jones, 1916; Chow, 1959; Henderson, 1966; Fread, 1975). This implies that the steady-flow rating curve is no longer sufficient and adequate to describe the real stage-discharge relationship. Recently, with a numerical study on the Po river, Di Baldassarre and Montanari (2009), showed that the use of the steady-flow rating curve may lead to large errors in discharge estimation when significant flood waves occur, which may be larger than 15% , and that another significant error is produced by the extrapolation of the rating curve beyond the range of measurements used for its derivation.

On top of the water balance error induced by the hysteretic effect and the extrapolation, another error occurs that may strongly affect the calibration and the verification of hydrological models: if calibration is made using discharge values derived from a steady-flow rating curve, then the estimated time of peak discharge will be wrong, because, under unsteady flow conditions, the peak discharge occurs before the maximum water stage, and this delay can be significant (several hours) in very mild river slope conditions.

Schmidt and Garcia (2003) describe different methods historically used to overcome this problem; these methods mainly consist in empirical adjustments of the rating curve, derived from experimental data, while, less frequently, especially in river reaches affected by backwater effects, estimations are adjusted using a reference value of water

surface slope, computed as the "fall", or difference in water level between the concerned section and a second reference section, where stage is known.

Aside from empirical methods, several formulas based on full or simplified dynamic flow equations have been developed, to account for the observed hysteresis in stage-discharge relationship. Most of these methods, estimate water surface slope using successive water stage measurements at the cross section where a rating curve is available, which is possible under kinematic approximating assumptions (Jones, 1916; Henderson, 1966). However, such formulations have been obtained only under restrictive hypothesis on flow and river bed geometry, which reduce the possibilities for their practical application (Schmidt and Yen, 2002; Perumal et al., 2004).

In this paper, an alternative methodology is introduced, which explicitly accounts for the longitudinal variation of the water surface slope, through the use of couples of simultaneous water stage measurements at two adjacent cross sections. Such procedure, which also requires the geometrical description of the two cross sections, allows for the application of the full dynamic flow equations without restrictive hypotheses.

The proposed methodology is fully described and compared, on the basis of several test bed experiments, to the wide variety of existing approaches that can be found in the literature.

2 Data and methods

As can be found in the literature, several authors have addressed the topic of unsteady flow discharge estimation (see Sect. 2.1). Different authors proposed original formulas or modifications to previous formulas, while others carried out comparisons or evaluations of existing formulas, using both numerical simulation or measured data in natural rivers. However, most of reviewed works lack of a comprehensive bibliographic research: the review of existing methodologies is either limited to a few well known methods, or referred to previous research made by the authors themselves. Consequently, a first aim of the present work was to review the available discharge estimation formulas.

2.1 Estimation methods based on stage measurements at a single section

The complete list of symbols used in the following equations can be found in Appendix A. The reviewed formulas are presented in chronological order; for reasons of space, the derivation of each formula from the dynamic flow equations have been omitted from the present paper but can be found in the referenced works.

2.1.1 Jones Formula

Among the formulas existing in literature, the Jones formula is, without any doubt, the most known one; according to Jones (1916), under kinematic wave approximating assumptions, unsteady-flow discharge can be computed as:

$$Q = Q_0 \left[1 + \frac{1}{S_0 c} \frac{\partial y}{\partial t} \right]^{1/2} \quad (1)$$

where the kinematic wave celerity c can be approximated from its definition:

$$c = -\frac{\partial Q}{\partial A} \cong -\frac{1}{B} \frac{\partial Q_0}{\partial z} \quad (2)$$

as can be found in Chow (1959) and Henderson (1966).

Since its publication, the Jones formula has been the subject of many research works, either as the starting point for obtaining more accurate equations, or for establishing a general applicability criterion; a number of these works are herein reviewed.

2.1.2 Henderson Formula

Henderson (1966) proposed to modify Eq. (1) through the introduction of a term which accounts for wave subsidence:

$$Q = Q_0 \left[1 + \frac{1}{S_0 c} \frac{\partial y}{\partial t} + \frac{2}{3r^2} \right]^{1/2} \quad (3)$$

where $r = S_0 / \left(\partial y / \partial x \right)$ is the ratio of the channel bed slope to the entering wave slope. According to Henderson (1966), the term r can be approximated from the characteristics of a typical flood event for the concerned reach; r is therefore given by the ratio of wave height to its half-length, the latter computed from the product of average wave celerity c and the time to peak stage (the typical wave is supposed to be kinematic).

2.1.3 Fread Formula

Starting from the hypothesis of Henderson formula (Eq. 3), Fread (1975) proposed an equation that allows to compute either the discharge Q or the water stage y as a function of time variation of the other variable:

$$Q - K \left[S_0 + \left[\frac{A}{MQ} + \left(1 - \frac{1}{M} \right) \frac{BQ}{gA^2} \right] \frac{\Delta z}{\Delta t} + \frac{1}{g} \frac{\Delta U}{\Delta t} + \frac{2S_0}{3r^2} \left(1 - \frac{BQ^2}{gA^3} \right) \right]^{1/2} = 0 \quad (4)$$

where $M = \frac{5}{3} + \frac{2A}{3B^2} \frac{\partial B}{\partial y}$ and r is the ratio of the channel bed slope to the entering wave slope, which can be computed using the following expression, similar to that proposed by Henderson for Eq. (3):

$$r = \frac{56\,200 \left(Q_p + Q_b \right)}{\left(h_p + h_b \right) \overline{A}} \times T_p \times S_0 \quad (5)$$

where Q_b and Q_p are the base and the peak discharge respectively; T_p is the time to peak (in days); h_b and h_p are the water stages corresponding to base and peak discharge; and $\overline{A}$ is the wetted area computed for the mean water stage.

The underlying hypothesis of Eq. (4) are correct in kinematic or quasi kinematic flow conditions, and in channel with approximately constant width (Fread, 1975); according to the author, the neglected diffusive terms may become significant for bed slopes less than 10^{-3} and wave rate of change greater than $3\,\mathrm{cm\,h^{-1}}$. Also note that Eq. (4) is implicit and therefore must be solved via iterative methods.

2.1.4 Marchi Formula

Marchi (1976) proposed an alternative version of an unsteady-flow rating curve using the following expression:

$$Q=Q_0+\frac{A}{2(m+1)BS_0}\left[1-m^2\frac{Q^2B}{gA^3}\right]\frac{\partial A}{\partial t} \tag{6}$$

where m is the exponent of the hydraulic radius in the friction law used. For instance, when using Chézy expression, $m=5/3$.

2.1.5 Lamberti and Pilati Formulas

Lamberti and Pilati (1990) developed two equations designed to compute the difference between steady and unsteady flow rating curves:

$$Q-Q_0=\frac{\partial z}{\partial t}T_1Bc \tag{7}$$

$$Q_{(t)}-Q_{0(t)}=\left(Q_{(t-1)}-Q_{0(t-1)}\right)\exp\left(-\frac{\Delta t}{T_2}\right)+T_1b\Delta t+T_1\left(1-\exp\left(-\frac{\Delta t}{T_2}\right)\right)(a-bT_2) \tag{8}$$

where $T_1=\frac{Q_0^2\left(1-Fr\left(1-cA/Q\right)^2\right)}{(Q+Q_0)S_zBc^2}$, $T_2=\frac{Q_0^2\left(1+Fr\left(c^2A^2/Q^2-1\right)\right)}{(Q+Q_0)S_zBc^2}$; a and b are first and second order incremental ratios defined as $a=\frac{Q_{0(t)}-Q_{0(t-2)}}{2\Delta t}$, $b=\frac{Q_{0(t)}-2Q_{0(t-1)}+Q_{0(t-2)}}{\Delta t^2}$; S_z is the water surface slope $\partial y/\partial x$, which can be approximated by bed slope S_0. Both formulas can be solved without iterations, using the terms T_1 and T_2 computed in the previous time step.

Equations (7) and (8) can be applied only in kinematic or quasi kinematic conditions, that is, in presence of narrow loops of the rating curve, with a maximum difference between unsteady and steady flow rating curve of about 10%. The authors provided a quantitative criterion to establish the ratio Q/Q_0 from channel and wave characteristics:

$$\frac{Q}{Q_0}=\frac{2T_1}{T_P} \tag{9}$$

where T_1 is the characteristic channel time, as defined for Eqs. (7) and (8), and T_p is the time to peak discharge.

2.1.6 Fenton Formula

Fenton (1999) proposed an extension of Jones formula which includes a diffusive term, according to the expression:

$$Q=Q_0\left[1+\frac{1}{S_0c}\frac{\partial y}{\partial t}-\frac{D}{S_0c^3}\frac{\partial^2 y}{\partial t^2}\right]^{1/2} \tag{10}$$

where D is the diffusion coefficient formulated as:

$$D=\frac{Q_0}{2BS_0} \tag{11}$$

2.1.7 Perumal Formulas

Perumal and Ranga Raju (1999), and Perumal et al. (2004) developed two modifications to the Jones formula; the first one has the following form:

$$Q=Q_0\left[1+\frac{1}{S_0c}\frac{\partial y}{\partial t}\left[1-m^2Fr^2P^2\left(\frac{\partial R/\partial y}{\partial A/\partial y}\right)^2\right]\right]^{1/2} \quad (12)$$

where m is the exponent of the hydraulic radius in the friction law used (as in Sect. 2.1.4 Eq. 6); the second equation is:

$$Q=\frac{Q_0}{\sqrt{2}}\left[1+\frac{1}{S_0c}\frac{\partial y}{\partial t}+\sqrt{\left(1+\frac{1}{S_0c}\frac{\partial y}{\partial t}\right)^2-\frac{2Q}{BS_0^2c^3}\frac{\partial^2 y}{\partial t^2}}\right]^{1/2} \quad (13)$$

Perumal and Moramarco (2005) reviewed a number of discharge estimation formulas; they pointed out that the Fenton formula (Eq. 10) can be regarded as a particular case of Eq. (13), while the Marchi Formula (Eq. 6) is an alternative writing of Eq. (12).

Perumal et al. (2004) also identified a criterion to establish the suitability of Eqs. (1), (12) and (13), as a function of bed and wave slopes; according to the authors, the estimation given by these methods may be considered good if the following condition holds:

$$\left|\frac{1}{S_0}\frac{\partial y}{\partial x}\right|\leq 0.5 \quad (14)$$

2.2 Approaches based on simultaneous stage measurements

The 1D shallow water momentum equation can be written in the form:

$$\frac{\partial z}{\partial x}+\frac{\beta U}{g}\frac{\partial U}{\partial x}+\frac{1}{g}\times\frac{\partial U}{\partial t}=-J \quad (15)$$

where $\frac{\partial z}{\partial x}=S_0+\frac{\partial y}{\partial x}$ is the water surface slope, composed by the channel bed slope and pressure force term; $\frac{\beta U}{g}\frac{\partial U}{\partial x}$ is the convective acceleration term; $\frac{1}{g}\times\frac{\partial U}{\partial t}$ is the local acceleration term; $J=\frac{Q^2}{K^2}$ is the hydraulic head slope while β is Boussinesq momentum coefficient and K the hydraulic conveyance.

Many authors presented a general discussion over the magnitude of the different terms composing Eq. (15) (see Henderson, 1966; Todini and Bossi, 1986; Lamberti and Pilati, 1996; Schmidt and Yen, 2003). In most rivers, during a flood event the local and the convective acceleration terms in Eq. (15), can be neglected because their values range from one tenth to one hundredth of the other terms appearing in the equation.

2.2.1 Chow Formula

By neglecting the convective and the local acceleration terms, a parabolic approximation of the full de Saint Venant equations can be obtained, which leads to Chow's (1959) formula:

$$Q=Q_0\left[1-\frac{1}{S_0}\frac{\partial y}{\partial x}\right]^{1/2} \quad (16)$$

It is important to notice that when the wave behaves as a kinematic wave, the longitudinal gradient of water stage can be directly related to the time derivative of the stage, by means of the kinematic celerity:

$$\frac{\partial y}{\partial x}=-\frac{1}{c}\frac{\partial y}{\partial t} \quad (17)$$

from which the Jones formula Eq. (1) can be derived.

Therefore, Jones formula can be regarded as an approximation of the parabolic assumptions used by Chow, which is valid when approaching the kinematic conditions expressed by Eq. (17).

2.2.2 The proposed DyRaC Formula

This simple parabolic approximation may be not suitable in non prismatic channels, where the effect of longitudinal variation of cross sections may be relevant so that the convective acceleration term in Eq. (15) becomes of the same magnitude of other terms (Schmidt and Yen, 2002). In this case the momentum balance equation Eq. (15) may be re-written as:

$$\frac{\partial\left(z+\frac{\beta\times Q^2}{2gA^2}\right)}{\partial x}+\frac{1}{g}\frac{\partial U}{\partial t}=-\frac{Q^2}{K^2} \tag{18a}$$

or, by neglecting the local acceleration term, as:

$$\frac{\partial\left(z+\frac{\beta\times Q^2}{2gA^2}\right)}{\partial x}=-\frac{Q^2}{K^2} \tag{18b}$$

as given in Aricò et al. (2008), Dottori et al. (2008).

The proposed approach neglects the continuity of mass equation between the two cross sections between which Eq. (18a, b) are applied, by assuming (1) that no significant discharge enters (or leaves) the reach between the two adjacent sections, and (2) the two cross sections are close enough to accept the hypothesis that $\partial Q/\partial x \cong 0$. On these grounds it is then possible to discretise Eq. (18a, b) between the upstream and the downstream cross sections, to obtain:

$$(z_u-z_d)+\frac{Q^2}{2g}\left(\frac{\beta_u}{A_u^2}-\frac{\beta_d}{A_d^2}\right)+\frac{(x_u-x_d)}{g}\frac{\partial U}{\partial t}=-\frac{(x_u-x_d)}{2}\left(\frac{1}{K_u^2}+\frac{1}{K_d^2}\right)\times Q^2 \tag{19a}$$

or, by neglecting the local acceleration term, as:

$$(z_u-z_d)+\frac{Q^2}{2g}\left(\frac{\beta_u}{A_u^2}-\frac{\beta_d}{A_d^2}\right)=-\frac{(x_u-x_d)}{2}\left(\frac{1}{K_u^2}+\frac{1}{K_d^2}\right)\times Q^2 \tag{19b}$$

which corresponds to the classical steady-flow backwater curve.

Please note that Eq. (21) is nothing else than the standard backwater curve used for the estimation of the water surface profile under steady (but non uniform) flow assumptions.

In Eq. (19a, b), the upstream and downstream conveyance values K_u and K_d can be computed assuming a constant energy slope along the section (Chow, 1958). Each cross section is divided in m subsections, each with conveyance K_j, and the total conveyance can be expressed as a function of the corresponding subsection conveyances, as:

$$K=\sum_{j=1}^{m}K_j=\frac{1}{n}\sum_{j=1}^{m}A_jR_j^{2/3} \tag{20}$$

Please note that the distance between the two adjacent sections must be sufficiently small to allow for the constant flow rate assumption to be realistic, but at the same time it must be sufficiently large to allow the difference in water stage to be grater than the measurement instrument sensitivity and the water elevation fluctuations.

Equation (21) can be solved explicitly with respect to Q, to give:

$$Q=\sqrt{\frac{2(z_u-z_d)}{(x_d-x_u)\left(\frac{1}{K_u^2}+\frac{1}{K_d^2}\right)-\frac{1}{g}\left(\frac{\beta_u}{A_u^2}-\frac{\beta_d}{A_d^2}\right)}} \tag{21}$$

Once the water levels in the upstream and downstream sections are measured, for a given roughness all the terms of Eq. (21) are known as a function of the water stage, therefore the equation can be used, similarly to a standard rating curve, to dynamically estimate the discharge values as a function of the water level as well as of the water surface slope, which continuously varies in time. This differs from the use of the classical steady-flow rating curve which only depends on the water depth by implicitly assuming an average, but constant in time, water surface slope. Therefore, due to its dynamic nature, this new approach will be called the Dynamic Rating Curve (DyRaC).

Whenever needed, namely when the local acceleration terms is not negligible, the DyRaC expression can be expanded by re-deriving it from Eq. (20), to give:

$$Q=\sqrt{\frac{2\left(z_u-z_d\right)-\frac{(x_d-x_u)}{g}\frac{\partial\bar{U}}{\partial t}}{(x_d-x_u)\left(\frac{1}{K_u^2}+\frac{1}{K_d^2}\right)-\frac{1}{g}\left(\frac{\beta_u}{A_u^2}-\frac{\beta_d}{A_d^2}\right)}} \quad (22)$$

where the time derivative $\partial U/\partial t$, can be approximated using the incremental ratio $\Delta\bar{U}/\Delta t$, where Δt is the sampling time step and $\bar{U}$ is the average velocity within the reach, which can be estimated as $\bar{U}=2Q/(A_u+A_d)$, which leads to:

$$Q\cong\sqrt{\frac{2\left(z_u-z_d\right)-\frac{(x_d-x_u)}{g}\frac{\bar{U}-\bar{U}_{t-\Delta t}}{\Delta t}}{(x_d-x_u)\left(\frac{1}{K_u^2}+\frac{1}{K_d^2}\right)-\frac{1}{g}\left(\frac{\beta_u}{A_u^2}-\frac{\beta_d}{A_d^2}\right)}} \quad (23)$$

where $\bar{U}_{t-\Delta t}=2Q_{t-\Delta t}/(A_u+A_d)_{t-\Delta t}$ is the average velocity computed at the previous time interval.

As opposed to Eq. (21), which is explicit in terms of discharge, Eq. (23) must be solved iteratively. This can be easily done using a simple Newton-Raphson approach, which converges to the required accuracy in a very limited number of iterations (~5–6).

Nonetheless, it will be shown that the results obtained using Eq. (23) are already adequate to accurately estimate the discharge in natural rivers.

2.3 Design and preparation of numerical experiments

As described in Sect. 2.1, many of the reviewed methods have been designed to provide flow discharge estimation in kinematic or quasi kinematic conditions; however, in such conditions, due to the limited amplitude of the unsteady flow loop, the formulas produce limited improvements with respect to what is obtained using the steady flow

rating curve. In most of the examples available in the literature, where discharge estimation formulas have been applied to natural rivers, flow conditions have always been chosen as quasi kinematic (Barbetta et al., 2002; Franchini and Ravagnani, 2007), whereas it would be of major interest to evaluate the quality of these expressions in non kinematic conditions, namely in the presence of wide loop rating curves, when the correcting term becomes substantial.

Therefore, a number of numerical experiments have been set up, to simulate a wide range of flow conditions over channels with different bed slope and geometry; these experiments are summarised in Table 1.

The values of bed slope used in the experiments vary from 10^{-3} (steep slope) to 2.5×10^{-5} (very mild slope), including the intermediate values of 5×10^{-4}, 2×10^{-4}, 10^{-4} and 5×10^{-5}; three types of wave have been used in the simulations: a fast wave with a rising time of 24 h and a peak discharge of 900 $m^3\,s^{-1}$, a medium wave with a rising time of 72 h and a peak discharge of 900 $m^3\,s^{-1}$ and a slow wave with a rising time of 168 h and a peak discharge of 10 000 $m^3 s^{-1}$. The choice of the bed slope values and the flood wave characteristics was made considering the results of numerical experiments carried out in previous works (Lamberti and Pilati, 1990; Perumal et al., 2004), in order to analyse not only typically kinematic or quasi kinematic flow conditions, but also to explore the range between kinematic and parabolic flow conditions.

In addition, the values of peak discharge, flood wave duration and channel geometry have been chosen as a function of bed slope values, in order to recreate flow conditions close to those which usually take place in natural rivers; for example, the channels with a bed slope of 5×10^{-5} and 2.5×10^{-5} have a section width of 400 m, much larger than the other channels with steeper bed slopes. The geometry of channels used in the numerical experiments is described more in detail in the sequel: cases from 1 to 8 relate to a the channel with rectangular cross sections and constant width; cases 9, 10 and 11 were introduced to assess the different expressions under variability of cross sections, and in particular case 9 is characterised by a cross section change from rectangular to trapezoidal, while cases 10 and 11 relate to a channel with irregular cross sections (Fig. 1), as one could expect from natural water courses.

The flood waves were generated in all the cases using the following expression:

$$Q(t) = Q_b + \left(Q_p - Q_b\right) \left[\frac{t}{T_p} \exp\left(1 - \frac{t}{T_p}\right)\right]^{\beta} \tag{24}$$

where Q_b is the base flow discharge (equal to $100\,\mathrm{m^3\,s^{-1}}$ in all cases), T_p the time to peak flow, Q_p the peak discharge and β a coefficient assumed to be equal to 16.

All the simulations were made using two well-known 1D hydraulic models, Hec-Ras (HEC, 2001) and Mike11 (DHI, 2003), in order to assess the results reliability. The results of the simulations using the two models always proved to be very similar both in terms discharge and stage values. These results were thus taken as the "true" values in order to assess the validity of the different formulas.

2.4 Formulas assessment

Before comparing among them the different approaches, the suitability of each method was assessed according to the criteria established by the authors or by other researchers in successive works.

The suitability of the Jones formula (Eq. 1) and of derived formulas presented by Perumal et al. (2004, Eqs. 6 and 7) has been evaluated using the criterion expressed by Eq. (14); applications show that these formulas should provide acceptable discharge values in cases 1 (fast wave over steep river bed slope), 2 (fast wave over medium river bed slope) and 4 (medium wave over medium-mild river bed slope); in cases 3 (fast wave over medium-mild river bed slope) and 6 (medium wave over mild river bed slope) the values obtained using criterion of Eq. (14) are occasionally greater than the threshold value, which means that estimation could be locally inexact, while in the remaining cases the results from formulas are expected to be not reliable. According to the analysis made by Perumal and Moramarco (2005), the same results may be considered valid also for Marchi and Fenton formulas (Eqs. 6 and 10).

An analogous procedure has been applied for the two equations developed by Lamberti and Pilati (1990, Eqs. 7 and 8); according to the criterion given by Eq. (9), these formulas should correctly estimate the discharge in cases 1, 2 and 4, while in remaining cases the results should be incorrect.

Fread (1975) stated that errors in his proposed formula (Eq. 4) may become significant for bed slopes smaller than 10^{-3} and wave rate of change greater than 3 cm per hour; this happens for cases 3 (fast wave over medium-mild river bed slope), 5 (fast wave over mild river bed slope) 7 and 8 (slow wave over very mild river bed slope). Theoretically, the same criterion can be applied to Henderson formula (Eq. 3), as Eq. (4) can be regarded as an extension of the latter.

Finally, the DyRaC formulas (Eqs. 21 and 23) are theoretically reliable under all flow conditions, in particular Eq. (23) is needed when the influence of the local acceleration term in Eq. (15) is not negligible, since this term may become significant in channels and rivers with very mild slopes subject to fast rising flood waves (hyperbolic flood wave conditions).

Several simulations were carried out in order to assess the relevance of the local acceleration term in the numerical experiments used to evaluate the effectiveness of the different equations. The results are presented in Fig. 2a and b, in terms of R, the ratio of the local acceleration term and the hydraulic head slope. The figures relate to cases 5 and 8, where R reaches its maximum values. As can be seen from the figures, the local acceleration term is always negligible, since R, which is plotted versus the hydraulic head slope reaches at most 1% of the latter. Therefore, due to the very small magnitude of the local acceleration term in all the reported experiments, which were chosen close to natural flood wave conditions in rivers, Eq. (23) was always used instead of Eq. (25) since it provides the same results, without requiring an iterative solution. Moreover, it should be noted that the waves simulated in cases 5 and 8 are significantly faster than flood waves generally taking place in natural rivers with similar bed slopes. For example, the bed slope of the final reach of the Po river in Italy is around 5×10^{-5}, while at the same time, the rising time of the flood waves is generally longer than one week (168 h), with a rate of change in stage of few cm h^{-1}. Therefore, although Eq. (23) can always be used when the inertial term becomes significant, it must be underlined that Eq. (21) can probably be applied, for practical operational purposes, on all types of natural rivers and under all flood conditions.

2.5 Operational estimation of discharge in natural rivers

Another topic of major relevance, is the reliability of the reviewed methods under operational conditions. Usually, the formulas presented in this paper were tested using high precision data from numerical or laboratory experiments, by assuming perfect water stage measurements, whereas operationally water stage measurements in natural rivers are generally affected by measurement errors (typically around ±1 cm) in terms of instrument precision, while local oscillations of the water surface can add additional uncertainty; as a consequence it is not possible to get a correct estimate of the real discharge using single instantaneous measurements.

An alternative methodology to provide reliable estimations can be applied by installing gauge stations with sensors capable to carry out a number of discharge estimates in a limited amount of time (few minutes), during which discharge can be considered as constant. This allows to iteratively compute the expected value of discharge $\mu(Q)$, using the following equation:

$$\mu_i(Q)=\frac{i-1}{i}\mu_{i-1}(Q)+\frac{1}{i}Q_i \qquad (25)$$

Where i is the number of measurements; Q_i is the i-th computed discharge value; $\mu_i(Q)$ and $\mu_{i-1}(Q)$ are the mean values computed using (i) and (i-1) measurements. The standard deviation of the computed values may also be estimated as:

$$\sigma_i(Q)=\sqrt{\mu_i\left(Q^2\right)-\mu_i^2(Q)} \qquad (26)$$

Where $\mu_i(Q^2)$ is the mean of square values of Q, estimated using the following recur-

sive equation:

$$\mu_i(Q^2)=\frac{i-1}{i}\mu_{i-1}(Q^2)+\frac{1}{i}Q_i^2 \qquad (27)$$

The accuracy of the estimated mean value of discharge is given by the standard deviation of the mean, defined as:

$$\sigma_i(\mu_i(Q))=\frac{\sigma_i(Q)}{\sqrt{i}} \qquad (28)$$

As can be seen from Eq. (28), the uncertainty on the estimation of Q reduces at each new measure, so that the procedure can be iterated until the error of estimation falls below a required precision.

The effectiveness of the proposed methodology needs then to be tested by showing the actual number of iterations required to reach an acceptable precision, which, for a practical use, must be limited. In the present paper, the methodology has been assessed by applying the following procedure: the reference values of water stage (computed by the hydraulic model as stated in Sect. 2.3) have been perturbed by adding a random error, computed using a Gaussian distribution with zero mean and a standard deviation of $\sqrt{5}$ cm, roughly comparable with an error deriving from the accuracy of water stage sensors (±1 cm) and from the water surface oscillations (±2 cm); for each time step a set of perturbed stage values was produced to simulate a series of continuous sensor measurements; then the procedure, starting from a minimum number of 10 and 20 couples of simultaneous stage measurements was iterated until the standard deviation $\sigma_i(\mu_i(Q))$ reached a value smaller than 5% respect to mean $\mu_i(Q)$. This was done by defining the following indicator $I_{\sigma/\mu}$, which was requested of being <0.05:

$$I_{\sigma/\mu}=\frac{\sigma_i(\mu_i(Q))}{\mu_i(Q)} \qquad (29)$$

3 Analysis of results

The estimated discharge values produced by the different formulas were evaluated by comparing the mean error and the error variance with respect to the discharge "true" values, namely the ones computed by using the hydraulic model (Sect. 2.3), taken as "true". In a first set of experiments (Sect. 3.1 and 3.2) the water stage measurements were considered as "perfect", namely not affected by measurement errors. The effect of measurement errors was then assessed and it is discussed in Sect. 3.3

3.1 Comparison on channels with prismatic constant section

Figures 3 and 4 show the mean error and the error variance of the succession in time of the discharge estimates produced by the alternative formulas for cases 3, 4, 5 and 6; the values obtained for the other cases were not represented to allow a clearer representation of results since the values were either very low (for cases 1 and 2) or very high (for cases 7 and 8) with respect to those presented in the two graphs. In addiction, some of the formulas were omitted because of the strong similarities existing among them: for instance Eq. (12) parameters almost coincide with those of Jones formula of Eq. (1), which was found also in a previous analysis work by Perumal et al. (2004). Moreover, in all cases from 1 to 8, the Chow and DyRaC formulas (Eqs. 16 and 21, respectively) gave the same results, which is not surprising given the use of prismatic cross sections.

As expected, the ability of the different equations to estimate discharge strongly depends on the channel and flood wave characteristics.

In cases 1 and 2 (fast wave over steep and medium river bed slope), the mean error is always below $2\,\mathrm{m}^3\,\mathrm{s}^{-1}$ for all the formulas and the percentage errors at peak are less than 1.2%, which means that they all very well reproduce the "true" values; however this is also true for the values given by the steady-flow rating curve; the discharge-level hydrograph (Fig. 5, left) and the comparison between steady and unsteady flow rating curves (Fig. 5, right) for case 2 shows the absence of a real loop, which implies that

flow conditions can be considered quasi kinematic.

In cases 3 (fast wave over medium-mild river bed slope), 4 (medium wave over medium-mild river bed slope), 6 (medium wave over mild river bed slope) and 7 (slow wave over mild river bed slope), the degree of accuracy is more variable since incoming waves become progressively steeper with respect to bed slope; nonetheless, it may be observed how DyRaC formula (Eq. 21) maintains a very low error rate, and that Perumal 2 (Eq. 13), Henderson (Eq. 3) and Fread (Eq. 4) formulas perform slightly better than other ones (see the hydrographs of case 3 presented in Fig. 6).

The performances given by Henderson and Fread formulas (Eqs. 3 and 4) are strongly dependent on the corrective coefficient r, which is a function of a so called "typical" or reference wave for the concerned reach (see Eq. 5); since it is not possible to set a reference wave for the channels used in the simulations, r was computed for each case from the incoming wave characteristics; such procedure, although it produces good results in theoretical cases, can only be applied in natural rivers to reconstruct the flood hydrograph after the event has passed, and not for an operational on-line discharge measurement.

The improved performance of Perumal 2 formula (Eq. 13) with respect to the others was also found by Perumal and Moramarco (2005), using similar numerical experiments.

In case 5 (fast wave over mild river bed slope), the accuracy of formulas based on single section measurements decreases significantly, as one can see from the observation of mean error values (Fig. 3) and from the hydrographs (Fig. 7); lastly, analysis of case 8 (slow wave over very mild river bed slope) shows that, in reaches with a very mild bed slope, none of the formulas using single water stage measurement is able to correctly estimate the discharge (Fig. 8). On the contrary, even in presence of fast flood waves, formulas using simultaneous couples of water stage measurements, like Chow (Eq. 16) and DyRaC (Eq. 21) formulas, provide accurate estimation, with a maximum error of the order of 1%.

3.2 Comparison on channels with spatially variable sections

The analysis of results in cases from 1 to 8 shows that the Chow and DyRaC formulas (Eqs. 16 and 21) provide almost coincident results when dealing with prismatic channels; however, as pointed out by Schmidt and Yen (2002), in natural rivers the Eq. (16) may become incorrect if the longitudinal section variation brings the convective acceleration terms to be relevant. The magnitude of this term has been evaluated using both a channel with varying prismatic sections (case 9) and a channel with irregular sections (cases 10 and 11); Fig. 9 illustrates flood hydrographs for cases 9 (left) and 11 (right) and, as can be seen, only the Jones, Chow and DyRaC formulas have been represented, along with the exact discharge and the steady flow rating curve.

In both cases, unlike the DyRaC formula (Eq. 21), the Chow approximation (Eq. 16) is not able to return the correct discharge hydrograph. Hence, it may be inferred that the parabolic approximation, which implies neglecting both the convective and the local acceleration terms, used in Eq. (16) can hardly be applied to discharge estimation in natural rivers unless, the concerned river reach is practically characterised by constant cross sections.

3.3 Measurement accuracy influence on discharge estimation

The methodology described in Sect. 2.5 has been applied to case 11, which uses irregular cross sections, to simulate a typical operational use of the DyRaC formula (Eq. 21). Figure 10 shows the resulting hydrograph compared to the "true" value and to the one derived from the steady-flow rating curve (top left); the values of $I_{\sigma/\mu}$, the cut-off indicator, obtained at each time step (top right), the error rate (bottom left) and the number of measurements needed to reach the required precision of 5% of $I_{\sigma/\mu}$ (bottom right). As can be seen, even when initialising the estimation process with a minimum number of samples (10) the required precision is automatically reached; only in a limited number of cases more measurements are necessary. Please note that in order to "filter" the water oscillations one is supposed to take measures at random

in time, with an average delay ranging from 1 to 5 s. Therefore, the obtained results imply that even in the worst cases one discharge value can be operationally estimated in a couple of minutes. Also note that the estimation accuracy can be improved by increasing the number of initial samples; the graphs in Fig. 11 show the results obtained using a minimum of 20 samples for each time step: as can be seen, the error rate significantly decreases with respect to the previous example shown in Fig. 10.

Although the described procedure should be operationally verified in real world applications, the results presented in this work are very promising and it is reasonable to believe that the DyRaC approach can be successfully applied in most of natural rivers.

4 Conclusions

Results obtained in the present work confirm the need to estimate discharge by means of expressions accounting for water surface slope, as stated by several authors (Henderson, 1966; Fenton, 2001; Schmidt and Garcia, 2003). Formulas not explicitly accounting for water surface slope can provide good estimations in kinematic or quasi cinematic conditions and, generally speaking, in channels with a steep bed slope (approximately 5×10^{-4} or greater), while they perform poorly in other conditions, especially in the presence of fast flood waves over mild bed slopes. In these cases, particularly in reaches with variable or irregular cross sections, it is necessary to directly measure the water surface slope and use a methodology like the proposed Dynamic Rating Curve. Results obtained by this procedure have proven to be accurate and reliable in all the numerical experiments; however, it is important to underline that the application of formulas using simultaneous stage measurements is slightly more demanding, in that, apart from the knowledge of the stage in two adjacent cross sections, it also requires the description of two rive cross section geometry and the use of a small piece of code.

Nonetheless, the DyRaC approach offers many advantages with respect to the use of the steady-flow rating curve: it allows to take into account the loops generated by the unsteady flow and the calibration procedure only requires the evaluation of roughness coefficient, thus eliminating the extrapolation errors. As opposed to the case of the steady-flow rating curve, a parameter of which controls the curvature of the rating curve, the parameter of DyRaC is the roughness coefficient, which more or less allows to move up and down the rating curve, while the curvature, which is fundamental when extrapolating beyond the range of measurements, is only driven by the cross section geometry, which is known.

Finally, as found in previous works (Dottori et al., 2008), the DyRaC methodology allows for an accurate discharge estimation also in sections affected by backwater effects, which influence is practically eliminated in the calibration phase.

Presently, a measurement instrument based on DyRaC is under development to be operationally installed and tested on several rivers showing different hydrological characteristics and conditions.

Appendix A

List of symbols used in equations

Q: discharge [$\mathrm{m^3\,s^{-1}}$];
Q_0: steady flow discharge, given by the steady-flow rating curve [$\mathrm{m^3\,s^{-1}}$];
y: water stage [m];
S_0: channel bed slope [-];
x: longitudinal distance along the reach [m];
z: water surface level [m];
B: cross section width at the water surface [m];
A: cross section area [$\mathrm{m^2}$];
P: cross section wetted perimeter [m];
R: cross section hydraulic radius [m];
K: cross section hydraulic conveyance [$\mathrm{m^3\,s^{-1}}$];
n: Manning roughness coefficient [$\mathrm{m^{-1/3}s}$];

g: acceleration due to gravity [$\mathrm{m\,s^{-2}}$];
U: mean velocity [$\mathrm{m\,s^{-1}}$];
Fr: Froude number [-];
c: kinematic wave celerity [$\mathrm{m\,s^{-1}}$];
Δt: time step of available data [s].

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References

Aricò, C., Tucciarelli, T., Dottori, F., Martina., M. L. V., and Todini, E.: Peak flow measurement in the Arno River by means of unsteady-state water level data analysis, Proc. International Conference on Fluvial Hydraulics (RiverFlow), Cesme-Izmir, Turkey, 3–5 September 2008.

Barbetta, S., Melone, F., Moramarco, T., and Saltalippi, C.: On Discharge Simulation from Observed Stage Hydrographs, Proc. IASTED International Conference, Crete, Greece, 25–28 June 2002.

Chow, V.-T.: Open Channel Hydraulics, Mc Graw Hill, Tokio, Japan, 680 pp., 1958.

Danish Hydraulic Institute (DHI): MIKE11 user's guide, Copenhagen, Denmark, 142 pp., 2003.

Di Baldassarre, G. and Montanari, A.: Uncertainty in river discharge observations: a quantitative analysis, Hydrol. Earth Syst. Sci. Discuss., 6, 39–61, 2009, http://www.hydrol-earth-syst-sci-discuss.net/6/39/2009/.

Dottori, F., Martina., M. L. V., and Todini, E.: Misure indirette di portata in alvei naturali, Atti del 31° Convegno Nazionale di Idraulica e Costruzioni Idrauliche, Perugia, Italy, 9–12 September 2008 (in Italian).

Fenton, J. D.: Calculating hydrographs from stage records, Proc. 28th IAHR Congress, Graz, Austria, 1999.

Fenton, J. D.: Rating Curves: Part 1 – Correction for Surface Slope, Proc. Conference on Hydraulics in Civil Engineering, Hobart, 309–317, Australia, 28–30 November 2001.

Franchini, M. and Ravagnani, F.: Costruzione della scala di deflusso in una sezione con sole misure di livello utilizzando le portate registrate a monte ed un modello diffusivo – convettivo, L'Acqua 5, 9–19, 2007 (in Italian).

Fread, D. L.: Computation of stage-discharge relationship affected by unsteady flow, Water Resources Bulletin, 11(2), 429–442, 1975.

Henderson, F. M.: Open channel flow, Macmilliam Series in Civil Engineering, Macmilliam eds., New York, USA, 522 pp., 1966.

Hydraulic Engineering Centre (HEC): Hydraulic Reference Manual, U.S. Army Corps of Engineers, Davis, California, USA, 377 pp., 2001.

Jones, B. E.: A method of correcting river discharge for a changing stage, U.S. Geological Survey Water Supply Paper, 375-E, 117–130, 1916.

Lamberti, P. and Pilati, S.: Quasi-kinematic flood wave propagation, Meccanica, 25, 107–114, 1990.

Lamberti, P. and Pilati, S.: Flood propagation models for real-time forecasting, J. Hydrol., 175, 239–265, 1996.

Marchi, E.: La Propagazione delle onde di piena, Atti Accademia Nazionale Lincei, 64, 594–602, 1976 (in Italian).

Nash, J. E. and Sutcliffe, J. V.: River flow forecasting through conceptual models part I – A discussion of principles, J. Hydrol., 10(3), 282–290, 1970.

Perumal, M. and Moramarco, T.: A reappraisal of discharge estimation methods using stage hydrographs, Proc. HYPESD International Conference; Roorkee, India, 23–25 February 2005, 105–116, 2005.

Perumal, M. and Ranga Raju, K. G.: Approximate convection-diffusion equations, J. Hydrol. Eng.-ASCE, 4(2), 160–164, 1999.

Perumal, M., Shrestha, K. B., and Chaube, U. C.: Reproduction of Hysteresis in Rating Curves, J. Hydrol. Eng.-ASCE, 130, 870–878, 2004.

Schmidt, A. R. and Garcia, M. H.: Theoretical Examination of Historical Shifts and Adjustments to Stage-Discharge Rating Curves, in Proc. EWRI World Water and Environmental Congress., Philadelphia, PA, USA, 23–26 June 2001.

Schmidt, A. R. and Yen, B. C.: Stage-Discharge Ratings Revisited, in Hydraulic Measurements and Experimental Methods, Proc. EWRI and IAHR Joint Conference, Estes Park, CO, USA, 28 July–1 August 2002.

Todini, E. and Bossi, A.: PAB (Parabolic and Backwater) an Unconditionally Stable Flood Routing Scheme Particularly Suited for Real Time Forecasting and Control, J. Hydrol. Eng.-ASCE, 24(5), 405–424, 1986.

Table 1. Characteristics of numerical experiments. In all the experiments, Manning's roughness has always been set equal to 0.035 $m^{-1/3}s$.

	Cross section geometry	Bed slope	Time to peak	Peak discharge ($m^3 s^{-1}$)
Case 1	Rectangular, 50 m width	10^{-3}	24 h	900
Case 2	Rectangular, 50 m width	5×10^{-4}	24 h	900
Case 3	Rectangular, 50 m width	2×10^{-4}	24 h	900
Case 4	Rectangular, 50 m width	2×10^{-4}	72 h	900
Case 5	Rectangular, 50 m width	10^{-4}	24 h	900
Case 6	Rectangular, 50 m width	10^{-4}	72 h	900
Case 7	Rectangular, 400 m width	5×10^{-5}	168 h	10 000
Case 8	Rectangular, 400 m width	2.5×10^{-5}	168 h	10 000
Case 9	Variable	5×10^{-4}	24 h	900
Case 10	Irregular	5×10^{-4}	24 h	900
Case 11	Irregular	2×10^{-4}	24 h	900

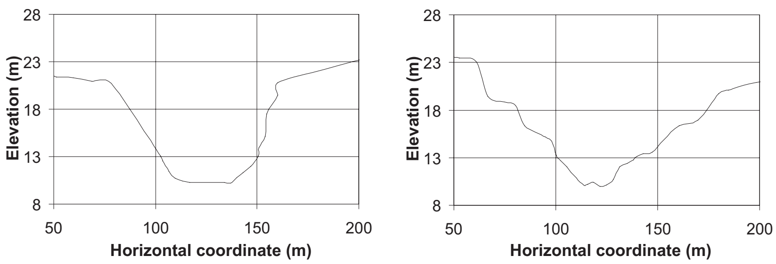

Fig. 1. Case 11: upstream (left) and downstream (right) cross sections in the channel reach where discharge has been estimated; distance between the two section is 1 km.

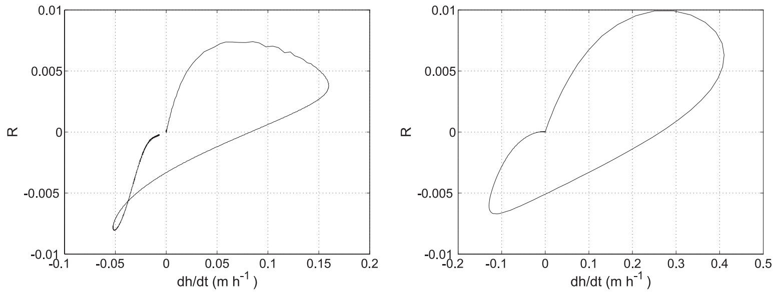

Fig. 2. Case 5 (left) and 8 (right); time evolution of R, the ratio between local acceleration term $(1/g)\times(\partial v/\partial t)$ and hydraulic head slope $\partial H/\partial x$, expressed as a function of the rate of change in stage $\partial h/\partial t$.

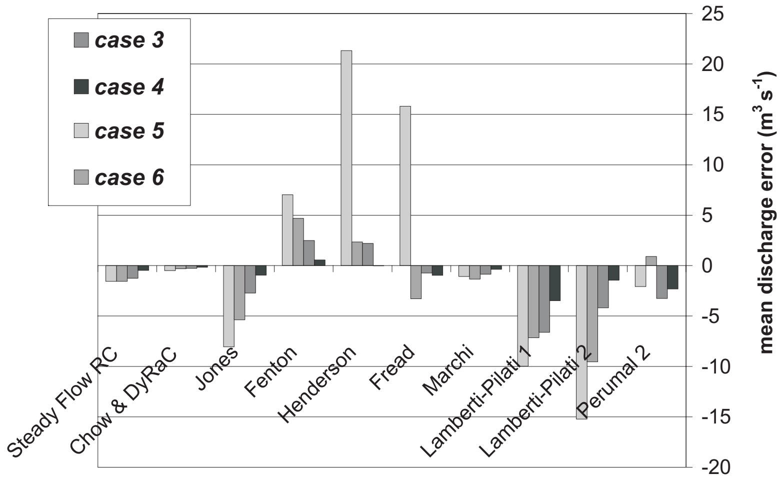

Fig. 3. Comparison of mean discharge error for cases 3, 4, 5 and 6.

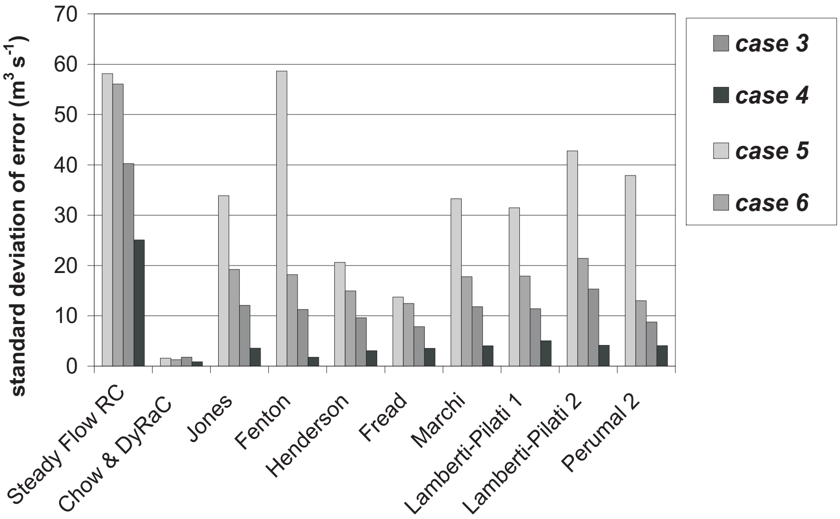

Fig. 4. Comparison of standard deviation of discharge estimation error for cases 3, 4, 5 and 6.

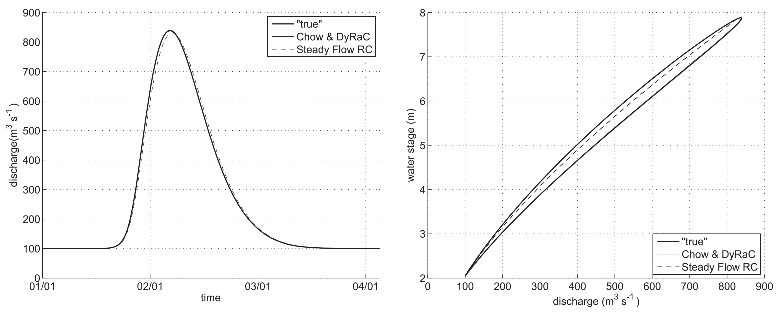

Fig. 5. Case 2 (channel with bed slope 5×10^{-4}, wave with a 24 h rising time period); left: estimated and "true" discharges hydrograph; right: estimated and "true" rating curves.

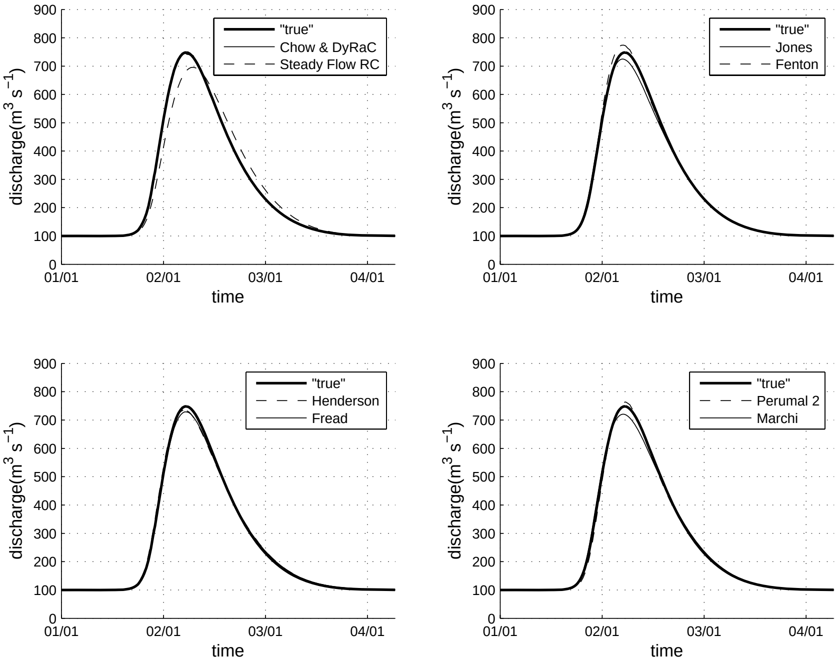

Fig. 6. Case 3 (channel with bed slope 2×10^{-4}, wave with a 24 h rising time period): estimated and "true" discharges hydrograph.

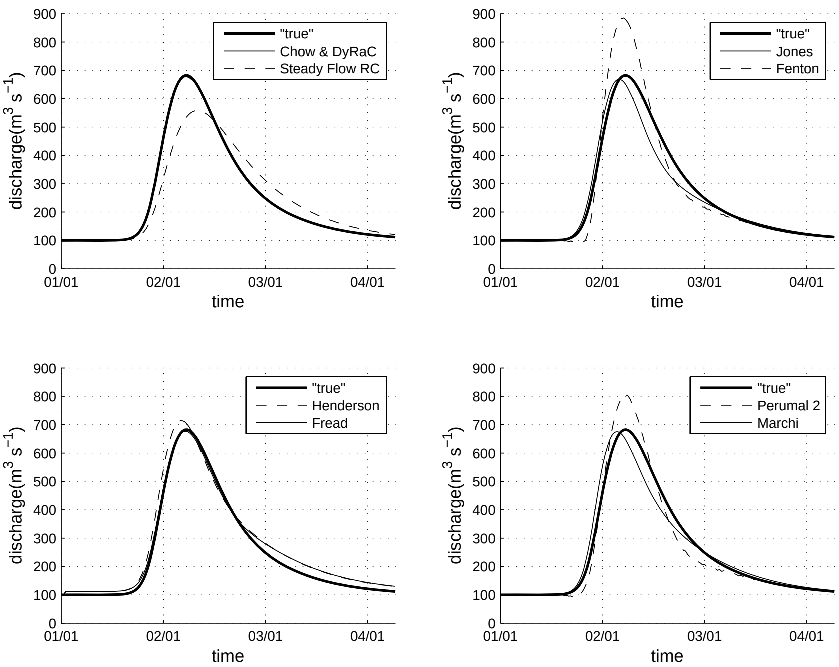

Fig. 7. Case 5 (channel with bed slope 10^{-4}, wave with a 24 h rising time period): estimated and "true" discharges hydrograph.

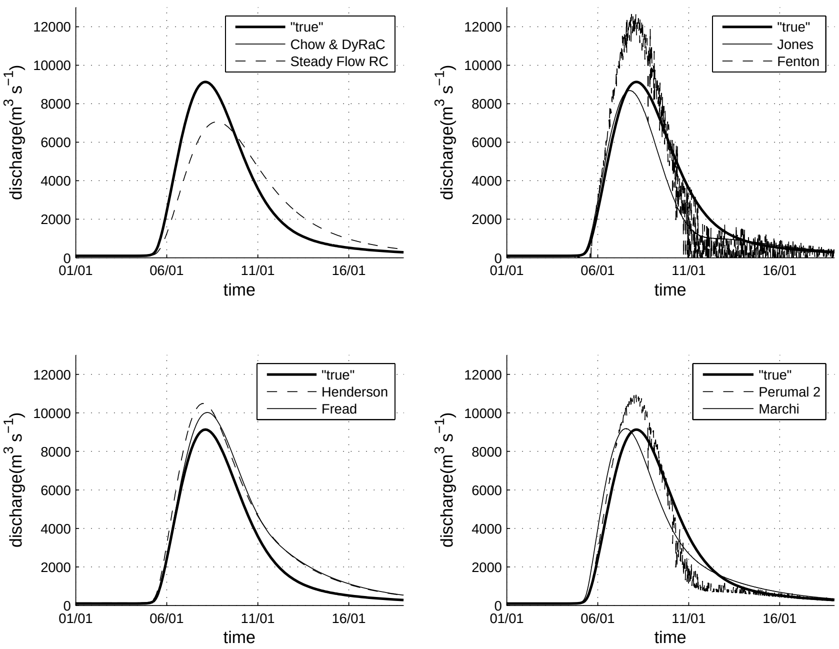

Fig. 8. Case 8 (channel with bed slope 2.5×10^{-5}, wave with a 168 h rising time period): estimated and "true" discharges hydrograph.

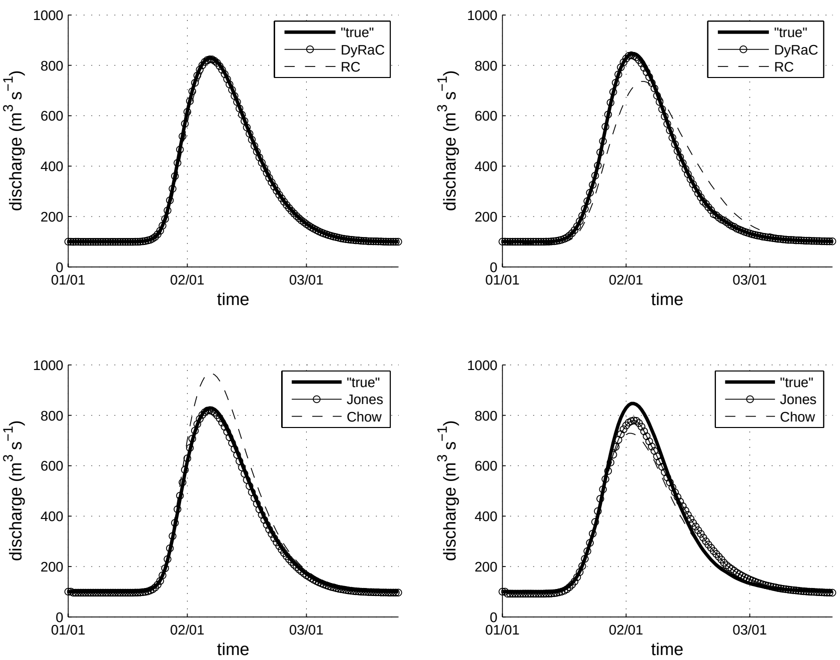

Fig. 9. On the left: estimated and "true" discharges hydrograph for case 9 (channel with bed slope 5×10^{-4}, and variable prismatic cross section); on the right: estimated and "true" discharges hydrograph for case 11 (channel with bed slope 2×10^{-4}, and variable irregular cross section).

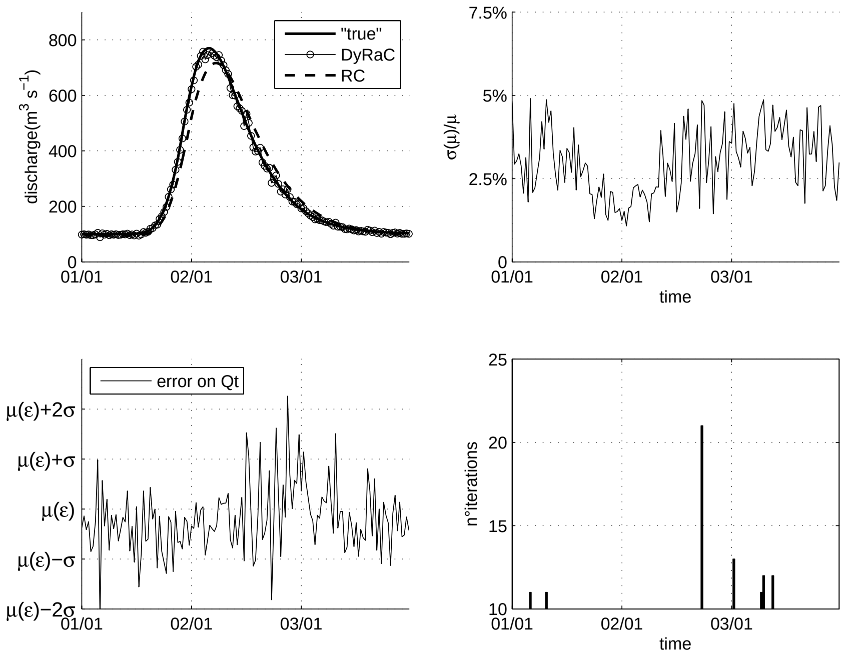

Fig. 10. Case 11 with error affected stage measurements; top-left: estimated and "true" discharges hydrograph; top-right: computed values of the cut-off indicator $I_{\sigma/\mu}$; bottom-left: normalised discharge estimation error (estimation error divided by "true" value); bottom-right: number of measurement samples needed to reach the requested accuracy: the minimum number for each time step is set to 10.

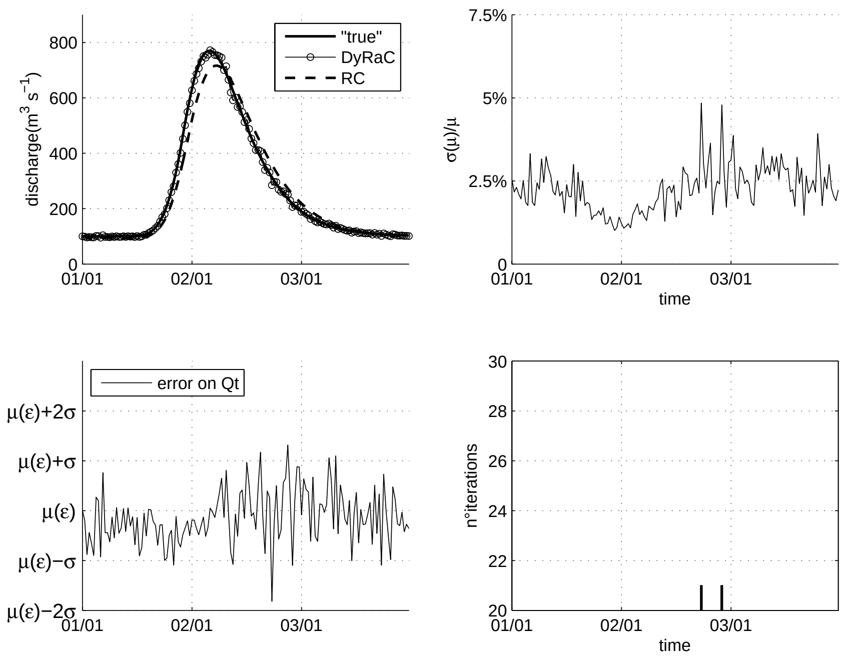

Fig. 11. Case 11 with error affected stage measurements; top-left: estimated and "true" discharges hydrograph; top-right: computed values of the cut-off indicator $I_{\sigma/\mu}$; bottom-left: normalised discharge estimation error (estimation error divided by "true" value); bottom-right: number of measurement samples needed to reach the requested accuracy: the minimum number for each time step is set to 20.