A General Analytical Model for Head Response to Oscillatory Pumping in Unconfined Aquifers: Consider the Effects of Delayed Gravity Drainage and Initial Condition

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Key points

1. An analytical model of the hydraulic head due to oscillatory pumping in unconfined aquifers is presented.
2. Head fluctuations affected by instantaneous and delayed gravity drainages are discussed.
3. The effect of initial condition on the phase of head fluctuation is analyzed.
4. The present solution agrees well to head fluctuation data taken from a field oscillatory pumping.
Abstract

Oscillatory pumping test (OPT) is an alternative to constant-head and constant-rate pumping tests for determining aquifer hydraulic parameters without net water extraction. There is a large number of analytical models presented for the analysis of OPT. The combined effects of delayed gravity drainage (DGD) and initial condition regarding the hydraulic head are commonly neglected in the existing models. This study aims to develop a new model for describing the hydraulic head fluctuation induced by OPT in an unconfined aquifer. The model contains a groundwater flow equation with the initial condition of static water table, Neumann boundary condition specified at the rim of a finite-radius well, and a free surface equation describing water table motion with the DGD effect. The solution of the model is derived by the Laplace transform, finite integral transform, and Weber transform. Sensitivity analysis is carried out for exploring head response to the change in each of hydraulic parameters. Results suggest the DGD reduces to instantaneous gravity drainage in predicting transient head fluctuation when dimensionless parameter $\alpha_1 = \varepsilon S_y b / K_z$ exceeds 500 with empirical constant $\varepsilon$, specific yield $S_y$, aquifer thickness $b$, and vertical hydraulic conductivity $K_z$. The water table can be regarded as a no-flow boundary when $\alpha_1 < 10^{-2}$. A pseudo-steady state model without initial condition causes a certain time shift from the actual transient model in predicting simple harmonic motion of head fluctuation during a late pumping period. In addition, the present solution agrees well to head fluctuation data observed at the Savannah River site.

KEYWORDS: oscillatory pumping test, analytical solution, free surface equation, delayed gravity drainage, initial condition
Notation and Abbreviation

\[ a = \frac{\sigma}{\mu} \]
\[ a_1, a_2 = \epsilon S_p b/K_z, \ a_1 \mu/\sigma \]
\[ b \] Aquifer thickness

DGD Delayed gravity drainage

\[ h \] Hydraulic head

\[ \bar{h} \] Dimensionless Hydraulic head, i.e., \( \bar{h} = (2\pi t K_r h)/|Q| \)

IGD Instantaneous gravity drainage

\[ K_r, K_z \] Aquifer horizontal and vertical hydraulic conductivities, respectively

LHS Left-hand side

\[ l \] Screen length, i.e., \( z_u - z_l \)

OPT Oscillatory pumping test

\[ P \] Period of oscillatory pumping rate

PSS Pseudo-steady state

\[ \bar{p} \] Dimensionless period, i.e., \( \bar{p} = (K_r P)/(S_s r_w^2) \)

\[ p \] Laplace parameter

\[ Q \] Amplitude of oscillatory pumping rate

RHS Right-hand side

\[ r \] Radial distance from the center of pumping well

\[ \bar{r} \] Dimensionless radial distance, i.e., \( \bar{r} = r/r_w \)

\[ r_w \] Radius of pumping well

SHM Simple harmonic motion

\[ S_s, S_y \] Specific storage and specific yield, respectively

\[ t \] Time since pumping

\[ \bar{t} \] Dimensionless pumping time, i.e., \( \bar{t} = (K_r t)/(S_s r_w^2) \)

\[ z \] Elevation from aquifer bottom

\[ z_l, z_u \] Lower and upper elevations of partial well screen, respectively

\[ \bar{z} \] Dimensionless elevation, i.e., \( \bar{z} = z/b \)

\[ \bar{z}_l, \bar{z}_u \] \( z_l/b, z_u/b \)

\[ \beta_n \] Roots of Eqs. (19)

\[ \gamma \] Dimensionless frequency of oscillatory pumping rate, i.e., \( S_s r_w^2 \omega/K_r \)

\[ \varepsilon \] Empirical constant associated with delayed gravity drainage

\[ \mu \] \( K_r r_w^2/K_r b^2 \)

\[ \sigma \] \( S_s/(S_s b) \)

\[ \omega \] Frequency of oscillatory pumping rate, i.e., \( \omega = 2\pi/P \)
1. Introduction

Numerous attempts have been made by researchers to the study of oscillatory pumping test (OPT) that is an alternative to constant-rate and constant-head pumping tests for determining aquifer hydraulic parameters (e.g., Vine et al., 2016; Christensen et al., 2017; Watlet et al., 2018). The concept of OPT was first proposed by Kuo (1972) in the petroleum literature. The process of OPT contains extraction stages and injection stages. The pumping rate, in other words, varies periodically as a sinusoidal function of time. Compared with traditional constant-rate pumping, OPT in contaminated aquifers has the following advantages: (1) low cost because of no disposing contaminated water from the well, (2) reduced risk of treating contaminated fluid, (3) smaller contaminant movement, and (4) stable signal easily distinguished from background disturbance such as tide effect and varying river stage (e.g., Spake and Mackley, 2011). However, the disadvantages of OPT includes the need of an advanced apparatus producing periodic rate and the problem of signal attenuation in remote distance from the pumping well. Oscillatory hydraulic tomography adopts several oscillatory pumping wells with different frequencies (e.g., Yeh and Liu, 2000; Cardiff et al., 2013; Zhou et al., 2016; Muthuwatta, et al., 2017). Aquifer heterogeneity can be mapped by analyzing multiple data collected from observation wells. Cardiff and Barrash (2011) reviewed articles associated with hydraulic tomography and classified them according to nine categories in a table.

Various groups of researchers have worked with analytical and numerical models for OPT; each group has its own model and investigation. For example, Black and Kipp (1981) assumed the response of confined flow to OPT as simple harmonic motion (SHM) in the absence of initial condition. Cardiff and Barrash (2014) built an optimization formulation strategy using the Black and Kipp analytical solution. Dagan and Rabinovich (2014) also assumed hydraulic head fluctuation as SHM for OPT at a partially penetrating well in unconfined aquifers. Cardiff et al. (2013) characterized aquifer heterogeneity using the finite element-based COMSOL software that adopts SHM hydraulic head variation for OPT. On the other hand, Rasmussen et
al. (2003) found hydraulic head response tends to SHM after a certain period of pumping time when considering initial condition prior to OPT. Bakhos et al. (2014) used the Rasmussen et al. (2003) analytical solution to quantify the time after which hydraulic head fluctuation can be regarded as SHM since OPT began. As mentioned above, most of the models for OPT assume hydraulic head fluctuation as SHM without initial condition, and all of them treat the pumping well as a line source with infinitesimal radius.

Field applications of OPT for determining aquifer parameters have been conducted in recent years. Rasmussen et al. (2003) estimated aquifer hydraulic parameters based on 1- or 2-hour period of OPT at the Savannah River site. Maineult et al. (2008) observed spontaneous potential temporal variation in aquifer diffusivity at a study site in Bochum, Germany. Fokker et al. (2012; 2013) presented spatial distributions of aquifer transmission and storage coefficient derived from curve fitting based on a numerical model and field data from experiments at the southern city-limits of Bochum, Germany. Rabinovich et al. (2015) estimated aquifer parameters of equivalent hydraulic conductivity, specific storage and specific yield at the Boise Hydrogeophysical Research Site by curve fitting based on observation data and the Dagan and Rabinovich (2014) analytical solution. They conclude the equivalent hydraulic parameters can represent the actual aquifer heterogeneity of the study site.

Although a large number of studies have been made in developing analytical models for OPT, little is known about the combined effects of delayed gravity drainage (DGD), finite-radius pumping well, and initial condition prior to OPT. Analytical solution to such a question will not only have important physical implications but also shed light on OPT model development. This study builds an improved model describing hydraulic head fluctuation induced by OPT in an unconfined aquifer. The model is composed of a typical flow equation with the initial condition of static water table, an inner boundary condition specified at the rim of the pumping well for incorporating finite-radius effect, and a free surface equation describing the motion of water table with the DGD effect. The analytical solution of the model
Based on the present solution, sensitivity analysis is performed to explore the hydraulic head in response to the change in each of hydraulic parameters. The effects of DGD and instantaneous gravity drainage (IGD) on the head fluctuations are compared. The quantitative criterion for treating the well radius as infinitesimal is discussed. The effect of the initial condition on the phase of head fluctuation is investigated. In addition, curve fitting of the present solution to head fluctuation data recorded at the Savannah River site is presented.

2. Methodology

2.1. Mathematical model

Consider an OPT in an unconfined aquifer illustrated in Fig. 1. The aquifer is of unbound lateral extent with a finite thickness $b$. The radial distance from the centerline of the well is $r$, an elevation from the impermeable bottom of the aquifer is $z$. The well with outer radius $r_w$ is screened from $z_u$ to $z_l$.

The flow equation describing spatiotemporal head distribution in aquifers can be written as:

$$K_r \left( \frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right) + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} \quad \text{for} \quad r_w \leq r < \infty, \quad 0 \leq z \leq b \quad \text{and} \quad t \geq 0 \quad (1)$$

where $h(r, z, t)$ is hydraulic head at location $(r, z)$ and time $t$; $K_r$ and $K_z$ are respectively the radial and vertical hydraulic conductivities; $S_s$ is the specific storage. Consider water table as a reference datum where the elevation head is set to zero; the initial condition is expressed as:

$$h = 0 \quad \text{at} \quad t = 0 \quad (2)$$

The rim of the wellbore is regarded as an inner boundary under the Neumann condition expressed as:

$$2\pi r_w K_r \frac{\partial h}{\partial r} = \begin{cases} Q \sin(\omega t) & \text{for} \quad z_l \leq z \leq z_u \\ 0 & \text{outside screen interval} \end{cases} \quad \text{at} \quad r = r_w \quad (3)$$

where $l = z_u - z_l$ is screen length; $Q$ and $\omega = 2\pi/P$ are respectively the amplitude and
frequency of oscillatory pumping rate (i.e., $Q\sin(\omega t)$) with a period $P$. Water table motion can be defined by Eq. (4a) for IGD (Neuman, 1972) and Eq. (4b) for DGD (Moench, 1995).

\[ K_x \frac{\partial h}{\partial x} = -S_y \frac{\partial h}{\partial x} \quad \text{at} \quad z = b \quad \text{for IGD} \tag{4a} \]

\[ K_x \frac{\partial h}{\partial x} = -\varepsilon \int_0^t \frac{\partial h}{\partial x} \exp(-\varepsilon(t - t')) \, dt' \quad \text{at} \quad z = b \quad \text{for DGD} \tag{4b} \]

where $S_y$ is the specific yield; $\varepsilon$ is an empirical constant. The impervious aquifer bottom is under the no-flow condition:

\[ \frac{\partial h}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{5} \]

The hydraulic head far away from the pumping well remains constant, written as

\[ \lim_{r \to \infty} h(r,z,t) = 0 \tag{6} \]

Define dimensionless variables and parameters as follows:

\[ \bar{h} = \frac{2\pi l K_f}{Q} h, \quad \bar{r} = \frac{r}{l}, \quad \bar{z} = \frac{z}{b}, \quad \bar{z}_1 = \frac{z_1}{b}, \quad \bar{z}_u = \frac{z_u}{b}, \quad \bar{t} = \frac{K_r}{\gamma S_f b^2} t, \quad \bar{P} = \frac{K_r}{\gamma S_f b^2} P \]

\[ \gamma = \frac{S_f r_u^2}{K_f}, \quad \mu = \frac{K_x r_u^2}{K_f b^2}, \quad \sigma = \frac{S_y}{S_f b}, \quad \alpha = \frac{\varepsilon S_f b}{K_f}, \quad \alpha_1 = \frac{\varepsilon S_f b}{K_x}, \quad \alpha_2 = \frac{\alpha \mu}{\sigma} \tag{7} \]

where the overbar stands for a dimensionless symbol. Note that the magnitude of $\alpha_1$ is related to the DGD effect (Moench, 1995) and $\gamma$ is a dimensionless frequency parameter. With Eq. (7), the dimensionless forms of Eqs. (1) - (6) become, respectively,

\[ \frac{\partial^2 \bar{h}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{h}}{\partial \bar{r}} + \mu \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{for} \quad 1 \leq \bar{r} < \infty, \quad 0 \leq \bar{z} < 1 \quad \text{and} \quad \bar{t} \geq 0 \tag{8} \]

\[ \bar{h} = 0 \quad \text{at} \quad \bar{r} = 0 \tag{9} \]

\[ \frac{\partial \bar{h}}{\partial \bar{r}} = \begin{cases} \sin(\gamma \bar{r}) & \text{for} \quad \bar{z}_1 \leq \bar{z} \leq \bar{z}_u \\ 0 & \text{outside screen interval} \end{cases} \quad \text{at} \quad \bar{r} = 1 \tag{10} \]

\[ \frac{\partial \bar{h}}{\partial \bar{r}} = -\alpha \frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 1 \quad \text{for IGD} \tag{11a} \]

\[ \frac{\partial \bar{h}}{\partial \bar{z}} = -a_1 \int_0^{\bar{t}} \frac{\partial \bar{h}}{\partial \bar{t'}} \exp(-a_2(\bar{t} - \bar{t'})) \, d\bar{t'} \quad \text{at} \quad \bar{z} = 1 \quad \text{for DGD} \tag{12b} \]

\[ \frac{\partial \bar{h}}{\partial \bar{z}} = 0 \quad \text{at} \quad \bar{z} = 0 \tag{13} \]

\[ \lim_{\bar{r} \to \infty} \bar{h}(\bar{r},\bar{z},\bar{t}) = 0 \tag{14} \]
Eqs. (8) – (13) represent the transient DGD model when excluding (11a) and transient IGD model when excluding (11b).

2.2. Transient solution for unconfined aquifer

The Laplace transform and finite integral transform are applied to solve Eqs. (8) - (13) (Liang et al., 2017). The former converts \( \bar{h}(\bar{r}, \bar{z}, \bar{t}) \) into \( \hat{h}(\bar{r}, \bar{z}, p) \), \( \partial \bar{h} / \partial \bar{t} \) in Eq. (8), (11) into \( p\bar{h} \), and \( \sin(\gamma \bar{r}) \) in Eq. (10) into \( \gamma / (p^2 + \gamma^2) \) with the Laplace parameter \( p \). The result of Eq. (8) in the Laplace domain can be written as

\[
\frac{\partial^2 \hat{h}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{h}}{\partial \bar{r}} + \mu \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} = p\hat{h} \tag{14}
\]

The transformed boundary conditions in \( r \) and \( z \) directions are expressed as

\[
\frac{\partial \hat{h}}{\partial \bar{r}} = \begin{cases} 
\frac{\gamma}{p^2 + \gamma^2} & \text{for } \bar{z}_l \leq \bar{z} \leq \bar{z}_u \\
0 & \text{outside screen interval}
\end{cases} \quad \text{at } \bar{r} = 1 \tag{15}
\]

\[
\frac{\partial \hat{h}}{\partial \bar{z}} = \begin{cases} 
-ap\hat{h} & \text{at } \bar{z} = 1 \text{ for IGD} \\
-\frac{a_p \hat{h}}{p + a_2} & \text{at } \bar{z} = 1 \text{ for DGD} \\
0 & \text{at } \bar{z} = 0 \tag{16a}
\end{cases}
\]

\[
\lim_{\bar{r} \to \infty} \hat{h}(\bar{r}, \bar{z}, p) = 0 \tag{18}
\]

The finite integral transform proposed by Latinopoulos (1985) is applied to Eqs. (14) - (17). The definition of the transform is given in Appendix A. Using the property of the transform converts \( \hat{h}(\bar{r}, \bar{z}, p) \) into \( \hat{h}(\bar{r}, \beta_n, p) \) and \( \partial^2 \hat{h} / \partial \bar{z}^2 \) in Eq. (14) into \( -\beta_n^2 \hat{h} \) with \( n \in (1, 2, 3, \ldots \infty) \) and \( \beta_n \) being the positive roots of the equation:

\[
\tan \beta_n = c / \beta_n \tag{19}
\]

where \( c = ap \) for IGD and \( a_p / (p + a_2) \) for DGD. The method to find the roots of \( \beta_n \) is discussed in section 2.3. Eq. (14) then becomes an ordinary differential equation (ODE) denoted as

\[
\frac{\partial^2 \hat{h}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{h}}{\partial \bar{r}} - \mu \beta_n^2 \hat{h} = p\hat{h} \tag{20}
\]

with the transformed Eqs. (18) and (15) written, respectively, as
\[ \lim_{\hat{r} \to \infty} \tilde{h}(\hat{r}, \beta_n, p) = 0 \quad (21a) \]

\[ \frac{\hat{h}}{\hat{r}} - \alpha p \hat{h} = \frac{\gamma F_\nu}{\beta_n (\nu^2 + \gamma^2)} \sin(\tilde{z}_u \beta_n) - \sin(\tilde{z}_l \beta_n) \quad \text{at} \quad \hat{r} = 1 \quad (21b) \]

where \( F = \sqrt{2(\beta_n^2 + c^2)/(\beta_n^2 + c^2 + \zeta)} \). Note that the transformation from Eq. (14) to (20) is applicable only for the no-flow condition specified at \( \tilde{z} = 0 \) (i.e., Eq. (17)) and third-type condition specified at \( \tilde{z} = 1 \) (i.e., Eq. (16a) or (16b)). Solve Eq. (20) with (21a) and (21b), and we can obtain:

\[ \tilde{h}(\hat{r}, \beta_n, p) = -\frac{\gamma F_0(\nu \lambda)(\sin(\tilde{z}_u \beta_n) - \sin(\tilde{z}_l \beta_n))}{\beta_n \nu K_1(\nu \lambda)(\nu^2 + \gamma^2)} \quad (22) \]

with

\[ \lambda = \sqrt{\nu + \mu \beta_n^2} \quad (23) \]

where \( K_0(-) \) and \( K_1(-) \) is the modified Bessel function of the second kind of order zero and one, respectively. Applying the inverse Laplace transform and inverse finite integral transform to Eq. (22) results in the transient solution expressed as

\[ \tilde{h}(\hat{r}, \beta_n, p) = \tilde{h}_{\exp}(\hat{r}, \tilde{z}, t) + \tilde{h}_{\text{SHM}}(\hat{r}, \tilde{z}, t) \quad (24a) \]

with

\[ \tilde{h}_{\exp}(\hat{r}, \tilde{z}, t) = \frac{-2\gamma}{\pi} \sum_{n=1}^{\infty} \int_0^\infty \cos(\beta_n \tilde{z}) \exp(p_0 \tilde{r}) \Im(e_1 e_2) \, d\zeta \quad (24b) \]

\[ \tilde{h}_{\text{SHM}}(\hat{r}, \tilde{z}, t) = A_t(\hat{r}, \tilde{z}) \cos\left(\gamma t - \phi_t(\hat{r}, \tilde{z})\right) \quad (24c) \]

\[ A_t(\hat{r}, \tilde{z}) = \sqrt{a_t(\hat{r}, \tilde{z})^2 + b_t(\hat{r}, \tilde{z})^2} \quad (24d) \]

\[ a_t(\hat{r}, \tilde{z}) = \frac{2}{\pi} \sum_{n=1}^{\infty} p_0 \cos(\beta_n \tilde{z}) \Im(e_1 e_2) \, d\zeta \quad (24e) \]

\[ b_t(\hat{r}, \tilde{z}) = \frac{2\gamma}{\pi} \sum_{n=1}^{\infty} \int_0^\infty \cos(\beta_n \tilde{z}) \Im(e_1 e_2) \, d\zeta \quad (24f) \]

\[ \phi_t(\hat{r}, \tilde{z}) = \cos^{-1}\left(b_t(\hat{r}, \tilde{z}) / A_t(\hat{r}, \tilde{z})\right) \quad (24g) \]

\[ e_1 = K_0(\lambda_0 \hat{r})(\sin(\tilde{z}_u \beta_n) - \sin(\tilde{z}_l \beta_n)) / (\beta_n \nu K_1(\nu \lambda_0)(\nu^2 + \gamma^2)) \quad (24h) \]

\[ e_2 = (\beta_n^2 + c_0^2) / (\beta_n^2 + c_0^2 + c_0) \quad (24i) \]

\[ p_0 = -\zeta - \mu \beta_n^2 \quad (24j) \]

\[ \lambda_0 = \sqrt{i} \quad (24k) \]
where \( c_0 = a \rho_0 \) for IGD and \( a_1 \rho_0/(\rho_0 + a_2) \) for DGD, \( i \) is the imaginary unit, and \( \text{Im}(-) \) is the imaginary part of a complex number. The detailed derivation of Eqs. (24a) – (24k) is presented in Appendix B. The first term on the right-hand side (RHS) of Eq. (24a) exhibits exponential decay due to the initial condition since pumping began; the second term defines SHM with amplitude \( \tilde{A}_t(\bar{r}, \bar{z}) \) and phase shift \( \phi_t(\bar{r}, \bar{z}) \) at a given point \((\bar{r}, \bar{z})\). The numerical results of the integrals in Eqs. (24b), (24e) and (24f) are obtained by the Mathematica \text{NIntegrate} \) function.

### 2.3. Calculation of \( \beta_n \)

The eigenvalues \( \beta_1, \ldots, \beta_n \), the roots of Eq. (19) with \( c = c_0 \) can be determined by applying the Mathematica function \text{FindRoot} based on Newton’s method with reasonable initial guesses. The roots are located at the intersection of the curves plotted by the RHS and left-hand side (LHS) functions of \( \beta_n \) in Eq. (19). The roots are very close to the vertical asymptotes of the periodical tangent function \( \tan \beta_n \). When \( c_0 = a \rho_0 \), the initial guess for each \( \beta_n \) can be considered as \( \beta_{0,n} + \delta \) where \( \beta_{0,n} = (2n-1)\pi/2 \), \( n \in (1,2,\ldots,\infty) \) and \( \delta \) is a small positive value set to \( 10^{-10} \) to prevent the denominator in Eq. (19) from zero. When \( c_0 = a_1 \rho_0/(\rho_0 + a_2) \), the initial guess is set to \( \beta_{0,n} - \delta \) for \( a_2 - \zeta \leq 0 \). There is an additional vertical asymptote at \( \beta_n = \sqrt{(a_2 - \zeta)/\mu} \) derived from the RHS function of Eq. (19) if \( a_2 - \zeta > 0 \). The initial guess is therefore set to \( \beta_{0,n} + \delta \) for \( \beta_{0,n} \) on the LHS of the asymptote and \( \beta_{0,n} - \delta \) for \( \beta_{0,n} \) on the RHS.

### 2.4. Transient solution for confined aquifer

When \( S_y = 0 \) (i.e., \( a = 0 \) or \( a_1 = 0 \)), Eq. (11a) or (11b) reduces to \( \partial \tilde{h}/\partial \bar{z} = 0 \) for no-flow condition at the top of the aquifer, indicating the unconfined aquifer becomes a confined one. Under this condition, Eq. (19) becomes \( \tan \beta_n = 0 \) with roots \( \beta_n = 0, \pi, 2\pi, \ldots, n\pi, \ldots, \infty \); Eq. (24i) reduces to \( \varepsilon_2 = 1 \); factor 2 in Eqs. (24b), (24e) and (24f) is replaced by unity for \( \beta_n = 0 \) and remains for the others. The analytical solution of the transient head for the confined aquifer can be expressed as Eqs. (24a) - (24k) with...
The resultant model is independent of \( \bar{r} \), indicating the analytical solution of \( H(\bar{r}, \bar{z}) \) is 2.5. Pseudo-steady state solution for unconfined aquifer

A pseudo-steady state (PSS) solution \( H_s \) accounts for SHM of head fluctuation after a certain period of pumping time and satisfies the following form (Dagan and Rabinovich, 2014)

\[
H_s(\bar{r}, \bar{z}, \bar{t}) = \text{Im}(\bar{H}(\bar{r}, \bar{z}) e^{i\gamma \bar{t}}) \tag{26}
\]

where \( \bar{H}(\bar{r}, \bar{z}) \) is a space function of \( \bar{r} \) and \( \bar{z} \). Define a PSS IGD model as Eqs. (8) - (13) excluding (9), (11b) and replacing \( \sin(\gamma \bar{t}) \) in (10) by \( e^{i\gamma \bar{t}} \). Substituting Eq. (26) and

\[
\frac{\partial^2 H}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial H}{\partial \bar{r}} + \frac{\mu}{\bar{r}^2} \frac{\partial^2 H}{\partial \bar{z}^2} = i\gamma \bar{H} \tag{27}
\]

\[
\frac{\partial H}{\partial \bar{r}} = \left\{ \begin{array}{ll} 1 & \text{for } \bar{z}_i \leq \bar{z} \leq \bar{z}_u \\ 0 & \text{outside screen interval} \end{array} \right. \text{ at } \bar{r} = 1 \tag{28}
\]

\[
\frac{\partial H}{\partial \bar{z}} = -i\alpha \gamma \bar{H} \text{ at } \bar{z} = 1 \text{ for IGD} \tag{29}
\]

\[
\frac{\partial \bar{H}}{\partial \bar{z}} = 0 \text{ at } \bar{z} = 0 \tag{30}
\]

\[
\lim_{\bar{t} \to \infty} \bar{H} = 0 \tag{31}
\]

The resultant model is independent of \( \bar{t} \), indicating the analytical solution of \( \bar{H}(\bar{r}, \bar{z}) \) is 11
tractable. Similarly, consider a PSS DGD model that equals the PSS IGD model but replaces (11a) by (11b). Substituting Eq. (26) into the result yields a model that depends on \( \bar{t} \), indicating the solution \( \bar{H}_s \) to the PSS DGD model is not tractable.

Taking the Weber transform to Eqs. (27) - (31) converts \( \bar{H} \) into \( \bar{H} \) and \( \partial^2 \bar{H}/\partial \bar{r}^2 \) into \(-\xi^2 \bar{H} - 2/(\pi \xi) \partial \bar{H}/\partial \bar{r} \big|_{\bar{r}=1} \). The result is expressed as

\[
\frac{\partial^2 \bar{H}}{\partial \bar{z}^2} - \lambda_w^2 \bar{H} = \begin{cases} \frac{2}{\pi \mu \xi} & \text{for } \bar{z}_l \leq \bar{z} \leq \bar{z}_u \\ 0 & \text{for } 0 \leq \bar{z} < \bar{z}_l \end{cases} \quad (32)
\]

\[
\frac{\partial \bar{H}}{\partial \bar{z}} = -i \gamma \bar{H} \quad \text{at } \bar{z} = 1 \quad (33)
\]

\[
\frac{\partial \bar{H}}{\partial \bar{z}} = 0 \quad \text{at } \bar{z} = 0 \quad (34)
\]

where \( \lambda_w^2 = (\xi^2 + i \gamma)/\mu \) and \( \xi \) is the Weber parameter. One can refer to Appendix C for the definition of the transform. Eq. (32) can be separated as

\[
\begin{align*}
\begin{cases}
\partial^2 \bar{H}_u/\partial \bar{z}^2 - \lambda_w^2 \bar{H}_u = 0 & \text{for } \bar{z}_u < \bar{z} \leq 1 \\
\partial^2 \bar{H}_m/\partial \bar{z}^2 - \lambda_w^2 \bar{H}_m = 2/(\pi \mu \xi) & \text{for } \bar{z}_l \leq \bar{z} \leq \bar{z}_u \\
\partial^2 \bar{H}_l/\partial \bar{z}^2 - \lambda_w^2 \bar{H}_l = 0 & \text{for } 0 \leq \bar{z} < \bar{z}_l 
\end{cases}
\end{align*}
\quad (35)
\]

with the continuity requirements:

\[
\begin{align*}
\begin{cases}
\bar{H}_m = \bar{H}_u & \text{at } \bar{z} = \bar{z}_u \\
\partial \bar{H}_m/\partial \bar{z} = \partial \bar{H}_u/\partial \bar{z} & \text{at } \bar{z} = \bar{z}_u 
\end{cases}
\end{align*}
\quad (36)
\]

\[
\begin{align*}
\begin{cases}
\bar{H}_l = \bar{H}_m & \text{at } \bar{z} = \bar{z}_l \\
\partial \bar{H}_l/\partial \bar{z} = \partial \bar{H}_m/\partial \bar{z} & \text{at } \bar{z} = \bar{z}_l 
\end{cases}
\end{align*}
\quad (37)
\]

Solving Eq. (35) with (33), (34), (36), and (37) results in

\[
\begin{align*}
\begin{cases}
\bar{H}_u = \bar{H}_p(c_1 \exp(\lambda_w \bar{z}) + c_2 \exp(-\lambda_w \bar{z})) & \text{for } \bar{z}_u < \bar{z} \leq 1 \\
\bar{H}_m = \bar{H}_p(c_3 \exp(\lambda_w \bar{z}) + c_4 \exp(-\lambda_w \bar{z}) - 1) & \text{for } \bar{z}_l \leq \bar{z} \leq \bar{z}_u \\
\bar{H}_l = \bar{H}_p c_5 (\exp(\lambda_w \bar{z}) + \exp(-\lambda_w \bar{z})) & \text{for } 0 \leq \bar{z} < \bar{z}_l 
\end{cases}
\end{align*}
\quad (38a)
\]

with

\[
\begin{align*}
c_1 &= -e^{-\lambda_w (\lambda_w - \alpha)}(\sinh(\bar{z}_l \lambda_w) - \sinh(\bar{z}_u \lambda_w))/D \\
c_2 &= -e^{\lambda_w (\lambda_w + \alpha)}(\sinh(\bar{z}_l \lambda_w) - \sinh(\bar{z}_u \lambda_w))/D \\
c_3 &= \frac{e^{-(1+z_i+z_2)\lambda_w}}{2D}(\alpha (\exp(2z_i\lambda_w) + e^{Z_i\lambda_w} - e^{Z_i+2Z_2\lambda_w}) + (\alpha - \lambda_w) e^{(Z_i+2Z_2)\lambda_w} + \\
\end{align*}
\quad (38b)
\]

\[
\begin{align*}
\end{align*}
\quad (38c)
\]

\[
\begin{align*}
\end{align*}
\quad (38d)
\]

\[
\begin{align*}
\end{align*}
\quad (38e)
\]
\[ \begin{align*}
262 \quad \lambda_w \left( e^{(2 \xi + z\bar{u}) \lambda_w} - e^{2 \xi \lambda_w} + e^{(2 \xi + z\bar{u}) \lambda_w} \right) & \quad (38d) \\
263 \quad c_4 = \frac{e^{-(1+\bar{u}+\bar{z}) \lambda_w}}{2D} \left( (\alpha - \lambda_w) e^{(2 \xi + 2 \bar{u}) \lambda_w} + (\alpha + \lambda_w) \left( e^{(2 \xi - 2 \bar{u}) \lambda_w} - e^{(2 \xi + 2 \bar{u}) \lambda_w} \right) \right) & \quad (38e) \\
264 \quad c_5 = \frac{1}{2} e^{-\left(1-\bar{z}\right)^2 \lambda_w} \left( e^{\xi \lambda_w} - e^{\xi \lambda_w} \right) \left( (\lambda_w - \alpha) e^{(2 \xi + 2 \bar{u}) \lambda_w} + (\lambda_w + \alpha) e^{2 \lambda_w} \right) & \quad (38f) \\
266 \quad \text{where } \alpha = i \gamma a, \; \bar{H}_P = 2/(\pi \mu \lambda_w^2) \text{ and } D = 2(\alpha \cosh \lambda_w + \lambda_w \sinh \lambda_w). \text{ The solution of } \bar{H} \\
267 \quad \text{given below can be obtained by the formula for the inverse Weber transform shown in Appendix C.} \\
268 \quad \Omega = \left( J_0(\xi \bar{r}) Y_1(\xi) - Y_0(\xi \bar{r}) J_1(\xi) \right) / (J_1^2(\xi) + Y_1^2(\xi)) & \quad (39b) \\
269 \quad \bar{H}(\bar{r}, \bar{z}) = \begin{cases} \\
0 & \text{for } \bar{z}_u < \bar{z} \leq 1 \\
\int_0^\infty \bar{H}_u \; \xi \; \Omega \; d\xi & \text{for } \bar{z}_l \leq \bar{z} \leq \bar{z}_u \\
\int_0^\infty \bar{H}_m \; \xi \; \Omega \; d\xi & \text{for } 0 \leq \bar{z} < \bar{z}_l \\
\int_0^\infty \bar{H}_l \; \xi \; \Omega \; d\xi & \text{for } \bar{z}_u < \bar{z} \leq 1 \\
0 & \text{for } \bar{z}_l < \bar{z} \leq \bar{z}_u \\
\end{cases} & \quad (39a) \\
270 \quad \text{with the Bessel functions of the first kind of order zero } J_0(\xi) \text{ and one } J_1(\xi) \text{ as well as the} \\
272 \quad \text{second kind of order zero } Y_0(\xi) \text{ and } Y_1(\xi). \text{ Note that the solution reduces to } \bar{H}(\bar{r}, \bar{z}) = \\
273 \quad \int_0^\infty \bar{H}_m \; \xi \; \Omega \; d\xi \text{ for a fully screened well when } \bar{z}_l = 0 \text{ and } \bar{z}_u = 1. \text{ With Eq. (26) and the} \\
274 \quad \text{formula of } e^{i\gamma \bar{r}} = \cos(\gamma \bar{r}) + i \sin(\gamma \bar{r}), \text{ the solution of } \bar{H}_s \text{ is expressed as} \\
275 \quad \bar{h}_s(\bar{r}, \bar{z}, \bar{t}) = \bar{A}_s(\bar{r}, \bar{z}) \cos(\gamma t - \phi_s(\bar{r}, \bar{z})) & \quad (40a) \\
276 \quad \bar{A}_s(\bar{r}, \bar{z}) = \sqrt{a_s(\bar{r}, \bar{z})^2 + b_s(\bar{r}, \bar{z})^2} & \quad (40b) \\
277 \quad a_s(\bar{r}, \bar{z}) = \text{Re}(\bar{H}(\bar{r}, \bar{z})) & \quad (40c) \\
278 \quad b_s(\bar{r}, \bar{z}) = \text{Im}(\bar{H}(\bar{r}, \bar{z})) & \quad (40d) \\
279 \quad \phi_s(\bar{r}, \bar{z}) = \cos^{-1} \left( \frac{b_s(\bar{r}, \bar{z})}{A_s(\bar{r}, \bar{z})} \right) & \quad (40e) \\
280 \quad \text{where Re(\cdot) is the real part of a complex number. Eq. (40a) indicates SHM for the response of} \\
281 \quad \text{the hydraulic head at any point to oscillatory pumping.} \\
282 \quad \text{2.6. Pseudo-steady state solution for confined aquifers} \\
283 \quad \text{Applying the finite Fourier cosine transform to the model, Eqs. (27) – (31) with } S_r = 0 \text{ (i.e.,} \\
284 \quad \alpha = 0 \text{) for the confined condition converts } \bar{H} \text{ into } \bar{h} \text{ and } \partial^2 \bar{H}/\partial \bar{z}^2 \text{ into } (m\pi)^2 \bar{H} \text{ with } m
being an integer from 0, 1, 2, … ∞. The result is written as

$$\frac{\partial^2 \hat{H}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{H}}{\partial \bar{r}} - \lambda_m^2 \hat{H} = 0$$  \tag{41}$$

$$\frac{\partial \hat{H}}{\partial \bar{r}} = \begin{cases} \frac{\bar{z}_u - \bar{z}_l}{m\pi} & \text{for } m = 0 \\ \frac{1}{m\pi} (\sin(\bar{z}_u m\pi) - \sin(\bar{z}_l m\pi)) & \text{for } m > 0 \end{cases} \text{ at } \bar{r} = 1$$  \tag{42}$$

$$\lim_{\bar{r} \to \infty} \hat{H} = 0$$  \tag{43}$$

where \( \lambda_m^2 = \gamma i + \mu(m\pi)^2 \); the result for \( m = 0 \) is derived by L'Hospital's. Solve Eq. (41) with (42) and (43), and we can have

$$\hat{H}(\bar{r}) = -\frac{\mathcal{K}_0(\rho \lambda_m)}{\lambda_m \mathcal{K}_1(\lambda_m)} \times \begin{cases} \frac{\bar{z}_u - \bar{z}_l}{m\pi} & \text{for } m = 0 \\ \frac{\cos(m\pi \bar{z})}{m\pi} (\sin(\bar{z}_u m\pi) - \sin(\bar{z}_l m\pi)) & \text{for } m > 0 \end{cases}$$  \tag{44}$$

After applying the inversion to Eq. (44) and the formula of \( e^{iy\bar{r}} = \cos(y\bar{r}) + i \sin(y\bar{r}) \), the solution of \( \tilde{h}_s \) for confined aquifers can be expressed as Eqs. (40a) - (40e) with \( H(\bar{r}, \bar{z}) \) replaced by

$$\hat{H}(\bar{r}, \bar{z}) = -2 \sum_{m=0}^{\infty} \frac{\mathcal{K}_0(\rho \lambda_m)}{\lambda_m \mathcal{K}_1(\lambda_m)} \times \begin{cases} 0.5(\bar{z}_u - \bar{z}_l) & \text{for } m = 0 \\ \frac{\cos(m\pi \bar{z})}{m\pi} (\sin(\bar{z}_u m\pi) - \sin(\bar{z}_l m\pi)) & \text{for } m > 0 \end{cases}$$  \tag{45}$$

For a fully screened well (i.e., \( \bar{z}_u = 1, \bar{z}_l = 0 \)), the first term of the series (i.e., \( m = 0 \)) remains and the others equal zero because of \( \sin(\bar{z}_u m\pi) - \sin(\bar{z}_l m\pi) = 0 \). The result is independent of dimensionless elevation \( \bar{z} \), indicating the confined flow is only horizontal.

### 2.7. Special cases of the present solution

Table 1 classifies the present solution (i.e., solution 1) and its special cases (i.e., solutions 2 to 6) according to transient or PSS flow, unconfined or confined aquifer, and IGD or DGD. Each of solutions 1 to 6 reduces to a special case for fully screened well. Existing analytical solutions can be regarded as special cases of the present solution as discussed in section 3.4 (e.g., Black and Kipp, 1981; Rasmussen et al., 2003; Dagan and Rabinovich, 2014).

### 2.8. Sensitivity analysis

Sensitivity analysis evaluates hydraulic head variation in response to the change in each of \( \mathcal{K}_\zeta, K_z, S_s, S_y, \omega, \) and \( \epsilon \). The normalized sensitivity coefficient can be defined as (Liou and Yeh, 2018).
where \( S_i \) is the sensitivity coefficient of the \( i \)th parameter; \( P_i \) is the magnitude of the \( i \)th input parameter; \( X \) represents the present solution in dimensional form. Eq. (46) can be approximated as

\[
S_i = P_i \frac{X(P_i + \Delta P_i) - X(P_i)}{\Delta P_i}
\]

where \( \Delta P_i \), a small increment, is chosen as \( 10^{-3} P_i \).

3. Results and Discussion

The following sections demonstrate the response of the hydraulic head to oscillatory pumping using the present solution. The default values in calculation are \( r = 0.05 \) m, \( z = 5 \) m, \( t = 0 \), \( b = 10 \) m, \( Q = 10^{-3} \) m\(^3\)/s, \( r_w = 0.05 \) m, \( z_u = 5.5 \) m, \( z_l = 4.5 \) m, \( K_r = 10^{-4} \) m/s, \( K_z = 10^{-5} \) m/s, \( S_s = 10^{-5} \) m\(^{-1} \), \( S_y = 10^{-4} \), \( \omega = 2\pi/30 \) s\(^{-1} \), and \( \varepsilon = 10^{-2} \) s\(^{-1} \). The corresponding dimensionless parameters and variables are \( \tilde{r} = 1 \), \( \tilde{z} = 0.5 \), \( \tilde{t} = 0 \), \( \tilde{z}_u = 0.55 \), \( \tilde{z}_l = 0.45 \), \( \gamma = 5.24 \times 10^{-5} \), \( \mu = 2.5 \times 10^{-6} \), \( a = 4 \times 10^5 \), \( a_1 = 1 \) and \( a_2 = 2.5 \times 10^{-6} \).

3.1. Delayed gravity drainage

Previous analytical models for OPT consider either confined flow (e.g., Rasmussen et al., 2003) or unconfined flow with IGD effect (e.g., Dagan and Rabinovich, 2014). Little attention has been given to the DGD effect. This section examines the relation between these three kinds of models. Figure 2 shows the curve of the dimensionless amplitude \( \tilde{A}_t \) at \((\tilde{r}, \tilde{z}) = (1, 1)\) of solution 1 versus the dimensionless parameter \( a_4 \) related to the effect. The transient head fluctuations are plotted by solution 1 with \( a_4 = 10^{-2} \), 1, 10, 500, solution 2 for IGD and solution 3 for confined flow. When \( 10^{-2} \leq a_4 \leq 500 \), the \( \tilde{A}_t \) gradually decreases with \( a_4 \) to the trough and then increases to the ultimate value of \( \tilde{A}_t = 1.79 \times 10^{-2} \). The DGD, in other words, causes an effect. When \( a_4 \leq 10^{-2} \), solutions 1 and 3 agree on the predicted heads, indicating the unconfined aquifer with the DGD effect behaves like confined aquifer and the
water table can be regarded as a no-flow boundary. When \( a_1 \geq 500 \), the head fluctuations predicted by solutions 1 and 2 are identical, indicating the DGD effect can be ignored and Eq. (4b) reduces to (4a) for the IGD condition.

### 3.2. Effect of finite radius of pumping well

Existing analytical models for OPT mostly treated the pumping well as a line source with infinitesimal radius (e.g., Rasmussen et al., 2003; Dagan and Rabinovich, 2014). The finite difference scheme for the model also treats the well as a nodal point by neglecting the radius. These will lead to significant error when a well has the radius ranging from 0.5 m to 2 m (Yeh and Chang, 2013). This section discusses the relative error in predicted amplitude defined as

\[
RE = \frac{|\tilde{A}_{D&\mathrm{R}} - \tilde{A}_t|}{\tilde{A}_t} \quad (48)
\]

where \( \tilde{A}_{D&\mathrm{R}} \) and \( \tilde{A}_t \) are the dimensionless amplitudes at \( \bar{r} = 1 \) (i.e., \( r = r_w \)) predicted by the Dagan and Rabinovich (2014) solution and the IGD solution 2. Note that their solution assumes infinitesimal radius of a pumping well and has a typo that the term \( e^{-(D_w + 1)} - e^{-D_w} \) should read \( e^{\beta(D_w + 1) - \beta D_w} \) (see their Eq. (25)). Figure 3 demonstrates the RE for different values of radius \( r_w \). The RE increases with \( r_w \) as expected. For case 1 of \( r_w = 0.1 \) m, both solutions agree well in the entire domain of \( 1 \leq \bar{r} \leq \infty \), indicating a pumping well with \( r_w \leq 0.1 \) m can be regarded as a line source. For the extreme case 2 of \( r_w = 1 \) m or case 3 of \( r_w = 2 \) m, the Dagan and Rabinovich solution underestimates the dimensionless amplitude for \( 1 \leq \bar{r} \leq 6 \) and agrees to the present solution for \( \bar{r} > 6 \). The REs for these two cases exceed 10%.

The effect of finite radius should therefore be considered in OPT models especially when observed hydraulic head data are taken close to the wellbore of a large-diameter well.

### 3.3. Sensitivity analysis

The temporal distributions of normalized sensitivity coefficient \( S_t \) defined as Eq. (47) with \( X = h_{\exp} \) of solution 1 are displayed in Fig. 4a for the response of exponential decay to the change in each of six parameters \( K_r, K_z, S_s, S_y, \omega \) and \( \varepsilon \). The exponential decay is very sensitive to variation in each of \( K_r, K_z, S_s \) and \( \omega \) because of \( |S_t| > 0 \). Precisely, a positive perturbation...
in $S_s$ produces an increase in the magnitude of $h_{\text{exp}}$ while that in $K_r$ or $K_z$ causes a decrease. In addition, a positive perturbation in $\omega$ yields an increase in $h_{\text{exp}}$ before $t = 1$ s and a decrease after that time. It is worth noting that $S_t$ for $S_y$ or $\varepsilon$ is very close to zero over the entire period of time, indicating $h_{\text{exp}}$ is insensitive to the change in $S_y$ or $\varepsilon$ and the subtle change of gravity drainage has no influence on the exponential decay. On the other hand, the spatial distributions of $S_t$ associated with the amplitude $A_t$ are shown in Fig. 4b in response to the changes in those six parameters. The $A_t$ is again sensitive to the change in each of $K_r$, $K_z$, $S_s$ and $\omega$ but insensitive to the change in $S_y$ or $\varepsilon$. The same result of $|S_t| \equiv 0$ for $S_y$ or $\varepsilon$ applies to any observation point under the water table (i.e., $\bar{z} < 1$), but $|S_t| > 0$ at the water table (i.e., $\bar{z} = 1$) shown in Fig. 4c. From those discussed above, we may conclude the changes in the four key parameters $K_r$, $K_z$, $S_s$ and $\omega$ significantly affect head prediction in the entire aquifer domain. The change in $S_y$ or $\varepsilon$ leads to insignificant variation in the predicted head below the water table and slight variation at the water table.

### 3.4. Transient head fluctuation affected by the initial condition

Figure 5 demonstrates head fluctuations predicted by DGD solution 1 and IGD solution 2 expressed as $h = h_{\text{exp}} + h_{\text{SHM}}$ for transient flow and by IGD solution as $h_s = \bar{A}_s \cos(\gamma t - \phi_s)$ for PSS flow. The transient head fluctuation starts from $h = 0$ at $t = 0$ and approaches SHM predicted by $h_{\text{SHM}}$ when $h_{\text{exp}} \equiv 0$ m after $t = 0.5P$ (i.e., $6 \times 10^4$). Solutions 1 and 2 agree to the $h$ predictions because the head at $\bar{z} = 0.5$ under the water table is insensitive to the change in $S_y$ or $\varepsilon$ as discussed in section 3.3. It is worth noting that the solution of Dagan and Rabinovich (2014) for PSS flow has a certain time shift from the $h_{\text{SHM}}$ of solution 2. This indicates the phase of their solution (i.e., 1.50 rad) should be replaced by the phase of solution 2 (i.e., $\phi_t = 1.64$ rad) so that their solution exactly fits the $h_{\text{SHM}}$ of solution 2.

Figure 6 displays head fluctuations predicted by transient solution 3 expressed as $h = h_{\text{exp}} + h_{\text{SHM}}$ and PSS solution 6 as $h_s = \bar{A}_s \cos(\gamma t - \phi_s)$ for partially-screened pumping.
well in panel (a) and full screen in panel (b). The Rasmussen et al. (2003) solution for transient 
flow predicts the same $\bar{h}$ as solution 3. The Black and Kipp (1981) for PPS flow also predict 
close $\bar{h}_{\text{SHM}}$ predictions of solution 3. The phase of solution 6 (i.e., $\phi_s = 1.50$ rad for panel 
(a) and 1.33 rad for (b)) should also be replaced by the phase of solution 3 (i.e., $\phi_t = 1.64$ 
rad for (a) and 1.81 rad for (b)) so that both solutions 3 and 6 agree to the SHM of head 
fluctuation. As concluded, excluding the initial condition with Eq. (26) for a PSS model leads 
to a certain time shift from the SHM of the head fluctuation predicted by the associated transient 
model while the transient and PSS models give the same SHM amplitude.

3.5. Application of the present solution to field experiment

Rasmussen et al. (2003) conducted field OPTs in a three-layered aquifer system containing one 
Surficial Aquifer, the Barnwell-McBean Aquifer in between and the deepest Gordon Aquifer 
at the Savannah River site. Two clay layers dividing these three aquifers may be regarded as 
impervious strata. For the OPT at the Surficial Aquifer, the formation has 6.25 m averaged 
thickness near the test site. The fully-screened pumping well has 7.6 cm outer radius. The 
pumping rate can be approximated as $Q \sin(\omega t)$ with $Q = 4.16 \times 10^{-4}$ m$^3$/s and $\omega = 2\pi$ h$^{-1}$. The 
distance from the pumping well is 6 m to the observation well 101D and 11.5 m to well 102D. 
The screen lengths are 3 m from the aquifer bottom for well 101D and from the water table for 
well 102D. For the OPT at the Barnwell-McBean Aquifer, the formation mainly consists of 
sand and fine-grained material. The pumping well has outer radius of 7.6 cm and pumping rate 
of $Q \sin(\omega t)$ with $Q = 1.19 \times 10^{-3}$ m$^3$/s and $\omega = \pi$ h$^{-1}$. The observation well 201C is at 6 m 
from the pumping well. The data of time-varying hydraulic heads at the observation wells (i.e., 
101D, 102D, 201C) are plotted in Fig. 7. One can refer to Rasmussen et al. (2003) for detailed 
description of the Savannah River site.

The aquifer hydraulic parameters are determined based on solutions 3 to 6 coupled with 
the Levenberg–Marquardt algorithm provided in the Mathematica function FindFit (Wolfram, 
1991). Solutions 4 and 5 are used to predict depth-averaged head expressed as
\( (z_u' - z_l')^{-1} \int_{z_l'}^{z_u'} h_s \, dz \) with the upper elevation \( z_u' \) and lower one \( z_l' \) of the finite screen of the observation well 101D or 102D at the Surficial Aquifer. Note that solutions 3 and 6 are independent of elevation because of the fully-screened pumping well. Define the standard error of estimate (SEE) as \( \text{SEE} = \sqrt{\frac{1}{M} \sum_{j=1}^{M} e_j^2} \) and the mean error (ME) as \( \text{ME} = \frac{1}{M} \sum_{j=1}^{M} e_j \) where \( e_j \) is the difference between predicted and observed hydraulic heads and \( M \) is the number of observation data (Yeh, 1987). The estimated parameters and associated SEE and ME are displayed in Table 2. The result shows the estimated \( S_r \) is very small, and the estimated \( T \) and \( S \) by solution 3 or 6 for confined flow are close to those by solution 4 or 5 for unconfined flow, indicating that the unconfined flow induced by the OPT in the Surficial Aquifer is negligibly small. Little gravity drainage due to the DGD effect appears with \( a_1 = 20 \) for wells 101D and 102D as discussed in section 3.1. Rasmussen et al. (2003) also revealed the confined behaviour of the OPT-induced flow in the Surficial Aquifer. The estimated \( S_j \) is one order less than the lower limit of the typical range of 0.01 ~ 0.3 (Freeze and Cherry, 1979), which accords with the findings of Rasmussen et al. (2003) and Rabinovich et al. (2015). Such a fact might be attributed to the problem of the moisture exchange limited by capillary fringe between the zones below and upper the water table. Several laboratory researches have confirmed an estimate of \( S_j \) at short period of OPT is much smaller than that determined by constant-rate pumping test (e.g., Cartwright et al., 2003; 2005). On the other hand, transient solution 3 gives smaller SEEs than PSS solution 6 for the Barnwell-McBean Aquifer and better fits to the observed data at the early pumping periods as shown in Fig. 7. From those discussed above, we may conclude the present solution is applicable to real-world OPT.

4. Concluding remarks

A variety of analytical models for OPT have been proposed so far, but little attention is paid to the joint effects of DGD, initial condition, and finite radius of a pumping well. This study develops a new model for describing hydraulic head fluctuation due to OPT in unconfined
aquifers. Static hydraulic head prior to OPT is regarded as an initial condition. A Neumann boundary condition is specified at the rim of a finite-radius pumping well. A free surface equation accounting for the DGD effect is considered as the top boundary condition. The solution of the model is derived by the Laplace transform, finite integral transform and Weber transform. The sensitivity analysis of the head response to the change in each of hydraulic parameters is performed. The observation data obtained from the OPT at the Savannah River site are analyzed by the present solution when coupling the Levenberg–Marquardt algorithm to estimate aquifer hydraulic parameters. Our findings are summarized below:

1. When $10^{-2} \leq a_1 \leq 500$, the effect of DGD on the head fluctuation should be considered. The amplitude of head fluctuation predicted by DGD solution 1 decreases with increasing $a_1$ to a certain trough and then increases to the amplitude predicted by IGD solution 2. When $a_1 > 500$, the DGD becomes IGD. Both solutions 1 and 2 predict the same head fluctuation. When $a_1 < 10^{-2}$, the DGD results in the water table under no-flow condition. Solution 1 for unconfined flow gives an identical head prediction to solution 3 for confined flow.

2. Assuming a large-diameter well as a line source with infinitesimal radius underestimates the amplitude of head fluctuation in the domain of $1 \leq \bar{r} \leq 6$ when the radius exceeds 80 cm, leading to relative error $RE > 10\%$ shown in Fig. 3. In contrast, the assumption is valid in predicting the amplitude in the domain of $\bar{r} > 6$ in spite of adopting a large-diameter well. When $r_w \leq 10$ cm (i.e., $RE < 0.45\%$), the well radius can be regarded as infinitesimal. The result is applicable to existing analytical solutions assuming infinitesimal radius and finite difference solutions treating the pumping well as a nodal point.

3. The sensitivity analysis suggests the changes in four parameters $K_r$, $K_z$, $S_y$ and $\omega$ significantly affect head prediction in the entire aquifer domain. The change in $S_y$ or $\epsilon$ causes insignificant variation in the head under water table but slight variation at the water table.
4. Analytical solutions for OPT are generally expressed as the sum of the exponential and harmonic functions of time (i.e., $h = \bar{h}_{\text{exp}} + \tilde{A}_t \cos(\gamma t - \phi_t)$) for transient solutions (e.g., solution 3) and harmonic function (i.e., $h_\text{s} = \tilde{A}_s \cos(\gamma t - \phi_s)$) for PSS solutions (e.g., solution 6). The latter assuming Eq. (26) without the initial condition produces a certain time shift from the SHM predicted by the $\bar{h}_{\text{SHM}}$. The phase $\phi_s$ should be replaced by $\phi_t$ so that $h_\text{s}$ and $\bar{h}_{\text{SHM}}$ are exactly the same.

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Yeh, T. C. J. and Liu, S. Y.: Hydraulic tomography: Development of a new aquifer test method,

Zhou, Y. Q., Lim, D., Cupola, F., and Cardiff, M.: Aquifer imaging with pressure waves

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**Appendix A: Finite integral transform**

Applying the finite integral transform to \( \hat{h} \) of the model, Eqs. (14) – (18), results in (Latinopoulos, 1985)

\[
\hat{h}(\beta_n) = \mathcal{F}\{\hat{h}(\bar{z})\} = \int_0^1 \hat{h}(\bar{z}) F \cos(\beta_n \bar{z}) d\bar{z} 
\]  
(A.1)

\[
F = \sqrt{\frac{2(\gamma^2 + \rho^2)}{\beta_n^2 + \rho^2}} 
\]  
(A.2)

where \( \beta_n \) is the root of Eq. (19). On the basis of integration by parts, one can write

\[
\mathcal{F}\left\{ \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} \right\} = \int_0^1 \left( \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} \right) F \cos(\beta_n \bar{z}) d\bar{z} = -\beta_n^2 \hat{h} 
\]  
(A.3)

Note that Eq. (A.3) is applicable only for the no-flow condition specified at \( \bar{z} = 0 \) (i.e., Eq. (17)) and third-type condition specified at \( \bar{z} = 1 \) (i.e., Eq. 16a or 16b). The formula for the inverse finite integral transform is defined as

\[
\hat{h}(\bar{z}) = \mathcal{F}^{-1}\{\hat{h}(\beta_n)\} = \sum_{n=1}^{\infty} \hat{h}(\beta_n) F \cos(\beta_n \bar{z}) 
\]  
(A.4)

**Appendix B: Derivation of Eqs. (24a) – (24k)**

On the basis of Eq. (A.4) and taking the inverse finite integral transform to Eq. (22), the Laplace-domain solution is obtained as

\[
\hat{h}(\bar{r}, \bar{z}, p) = 2 \sum_{n=1}^{\infty} \hat{h}(\bar{r}, \beta_n, p) \cos(\beta_n \bar{z}) 
\]  
(B.1)

with

\[
\hat{h}(\bar{r}, \beta_n, p) = \hat{h}_1(p) \cdot \hat{h}_2(p) 
\]  
(B.2)

\[
\hat{h}_1(p) = \frac{\psi}{(\rho^2 + \rho^2)} 
\]  
(B.3)

\[
\hat{h}_2(p) = -\varphi_1 \varphi_2 
\]  
(B.4)

\[
\varphi_1 = K_0(\bar{r} \lambda) (\sin(\bar{z}_1 \beta_n) - \sin(\bar{z}_2 \beta_n)) / (\beta_n^2 \mathcal{K}_1(\lambda)) 
\]  
(B.5)
\[ \varphi_2 = \frac{\beta_n^2 + c^2}{\beta_n^2 + c^2 + c} \quad \text{(B.6)} \]

where \( \lambda \) is defined in Eq. (23). Using the Mathematica function `InverseLaplaceTransform`, the inverse Laplace transform for \( \tilde{h}_1(p) \) in Eq. (B.3) is obtained as

\[ \tilde{h}_1(t) = \sin(\pi t) \quad \text{(B.7)} \]

The inverse Laplace transform for \( \tilde{h}_2(p) \) in Eq. (B.4) is written as

\[ \tilde{h}_2(t) = \frac{1}{2\pi i} \lim_{\rho \to \infty} \int_{\rho - i\infty}^{\rho + i\infty} \tilde{h}_2(p) e^{\pi t} dp \quad \text{(B.8)} \]

where \( \rho \) is a real number being large enough so that all singularities are on the LHS of the straight line from \((\rho, -i\infty)\) to \((\rho, i\infty)\) in the complex plane. The integrand \( \tilde{h}_2(p) \) is a multiple-value function with a branch point at \( p = -\mu \beta_n^2 \) and a branch cut from the point along the negative real axis. In order to reduce \( \tilde{h}_2(p) \) to a single-value function, we consider a modified Bromwich contour that contains a straight line \( \overline{AB}, \overline{CD} \) right above the branch cut and \( \overline{EF} \) right below the branch cut, a semicircle with radius \( R \), and a circle \( \overline{DE} \) with radius \( r' \) in Fig. A1. According to the residual theory, Eq. (B.8) may be expressed as

\[ \tilde{h}_2(t) + \lim_{r' \to 0} \frac{1}{2\pi i} \int_{R - \infty}^{R + \infty} \tilde{h}_2(p) e^{\pi t} dp + \int_{D}^{E} \tilde{h}_2(p) e^{\pi t} dp + \int_{F}^{E} \tilde{h}_2(p) e^{\pi t} dp + \int_{E}^{A} \tilde{h}_2(p) e^{\pi t} dp = 0 \quad \text{(B.10)} \]

where zero on the RHS is due to no pole in the complex plane. The integrations for paths \( \overline{BA} \) (i.e. \( \int_{B}^{C} \tilde{h}_2(p) e^{\pi t} dp + \int_{F}^{A} \tilde{h}_2(p) e^{\pi t} dp \)) with \( R \to \infty \) and \( \overline{DE} \) (i.e. \( \int_{D}^{E} \tilde{h}_2(p) e^{\pi t} dp \)) with \( r' \to 0 \) equal zero. The path \( \overline{CD} \) starts from \( p = -\infty \) to \( p = -\mu \beta_n^2 \) and \( \overline{EF} \) starts from \( p = -\mu \beta_n^2 \) to \( p = -\infty \). Eq. (B.10) therefore reduces to

\[ \tilde{h}_2(t) = -\frac{1}{2\pi i} \left( \int_{-\infty}^{-\mu \beta_n^2} \tilde{h}_2(p^+) e^{\pi t} dp + \int_{-\mu \beta_n^2}^{-\infty} \tilde{h}_2(p^-) e^{\pi t} dp \right) \quad \text{(B.11)} \]

where \( p^+ \) and \( p^- \) are complex numbers right above and below the real axis, respectively.

Consider \( p^+ = \zeta e^{i\pi} - \mu \beta_n^2 \) and \( p^- = \zeta e^{-i\pi} - \mu \beta_n^2 \) in the polar coordinate system with the origin at \((-\mu \beta_n^2, 0)\) in the complex plane. Eq. (B.11) then becomes
\[ \tilde{h}_2(\xi) = \frac{-1}{2\pi i} \int_0^\infty \tilde{h}_2(p^+) e^{p^+ \xi} dp - \tilde{h}_2(p^-) e^{p^- \xi} d\zeta \]  
\hspace{1cm} (B12)

where \( p^+ \) and \( p^- \) lead to the same result of \( p_0 = -\zeta - \mu \beta_n^2 \) for a given \( \zeta \); \( \lambda = \sqrt{p + \mu \beta_n^2} \) equals \( \lambda_0 = \sqrt{\xi} i \) for \( p = p^+ \) and \( -\lambda_0 \) for \( p = p^- \). Note that \( \tilde{h}_2(p^+) e^{p^+ \xi} \) and \( \tilde{h}_2(p^-) e^{p^- \xi} \) are in terms of complex numbers. The numerical result of the integrand in Eq. (B.12) must be a pure imaginary number that is exactly twice of the imaginary part of a complex number from \( \tilde{h}_2(p^+) e^{p^+ \xi} \) with \( p^+ = p_0 \) and \( \lambda = \lambda_0 \). The inverse Laplace transform for \( \tilde{h}_2(p) \) can be written as

\[ \tilde{h}_2(\xi) = \frac{-1}{\pi} \int_0^\infty \text{Im} \left( \varphi_1 \varepsilon_2 e^{p \xi} \right) d\xi \]  
\hspace{1cm} (B13)

where \( p = p_0 \); \( \lambda = \lambda_0 \); \( \varphi_1 \) and \( \varepsilon_2 \) are respectively defined in Eqs. (B.5) and (24i).

According to the convolution theory, the inverse Laplace transform for \( \tilde{h}(\tilde{\tau}, \beta, p) \) is

\[ \tilde{h}(\tilde{\tau}, \beta, p) = \int_0^\tilde{\tau} \tilde{h}_2(\tau) \tilde{h}_1(\tilde{\tau} - \tau) d\tau \]  
\hspace{1cm} (B14)

where \( \tilde{h}_1(\tilde{\tau} - \tau) = \sin(\gamma(\tilde{\tau} - \tau)) \) based on Eq. (B.7); \( \tilde{h}_2(\tau) \) is defined in Eq. (B.13) with \( \tilde{\tau} = \tau \). Eq. (B.14) can reduce to

\[ \tilde{h}(\tilde{\tau}, \beta, p) = \frac{-1}{\pi} \int_0^\infty \text{Im} \left( \varphi_1 \varepsilon_2 \frac{(p \varepsilon_2 \gamma e^{p \xi} - \gamma \cos(\gamma \xi) - p_0 \sin(\gamma \xi))}{p_0^2 + \gamma^2} \right) d\xi \]  
\hspace{1cm} (B15)

Substituting \( \tilde{h}(\tilde{\tau}, \beta, p) = \tilde{h}(\tilde{\tau}, \beta, \tilde{\tau}) \) and \( \tilde{h}(\tilde{\tau}, \tilde{z}, p) = \tilde{h}(\tilde{\tau}, \tilde{z}, \tilde{\tau}) \) into Eq. (B.1) and rearranging the result leads to

\[ \tilde{h}(\tilde{\tau}, \tilde{z}, \tilde{\tau}) = \frac{-2}{\pi} \sum_{n=1}^\infty \int_0^\infty \cos(\beta_n \tilde{z}) \text{Im}(\varepsilon_1 \varepsilon_2 \gamma e^{p_0 \xi}) d\xi + \]  
\hspace{1cm} (B16)

\[ \frac{2}{\pi} \sum_{n=1}^\infty \int_0^\infty \cos(\beta_n \tilde{z}) \text{Im}(\varepsilon_1 \varepsilon_2 \gamma \cos(\gamma \xi) + p_0 \sin(\gamma \xi)) d\xi \]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are defined in Eqs. (24h) and (24i); the first RHS term equals \( \tilde{h}_{\text{exp}}(\tilde{\tau}, \tilde{z}, \tilde{\tau}) \) defined in Eq. (24b); the second term is denoted as \( \tilde{h}_{\text{SHM}}(\tilde{\tau}, \tilde{z}, \tilde{\tau}) \) defined in Eq. (24c). Finally, the complete solution is expressed as Eqs. (24a) – (24k).

**Appendix C: Weber transform**

Applying the Weber transform to \( H \) of the model, Eqs. (27) – (31), yields
\[ \bar{H}(\zeta) = \mathcal{W}(\bar{H}) = \int_{1}^{\infty} \bar{H} \; \bar{r} \; \Omega \; dr \]  

(C1)

where \( \Omega \) is defined in Eq. (39b). With the integration by parts, the transform has the property

\[ \mathcal{W}\left\{ \frac{\partial^2 \bar{H}}{\partial \zeta^2} + \frac{1}{\bar{r}} \frac{\partial \bar{H}}{\partial \bar{r}} \right\} = -\xi^2 \bar{H} - \frac{2}{\pi \xi} \frac{dH}{dr} \bigg|_{r=1} \]  

(C2)

where the second RHS term represents the Neumann boundary condition Eq. (28). The formula for the inversion can be written as

\[ \bar{H} = \mathcal{W}^{-1}\{\bar{H}\} = \int_{0}^{\infty} \bar{H} \; \xi \; \Omega \; d\xi \]  

(C3)
Table 1. The present solution and its special cases

<table>
<thead>
<tr>
<th>Well screen</th>
<th>Transient flow</th>
<th>Pseudo-steady state flow</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unconfined aquifer</td>
<td>Confined aquifer</td>
</tr>
<tr>
<td>Partial</td>
<td>Solutions 1 and 2</td>
<td>Solution 3</td>
</tr>
<tr>
<td>Full</td>
<td>Solutions 1 and 2&lt;sup&gt;a&lt;/sup&gt;</td>
<td>Solution 3&lt;sup&gt;b&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

Solution 1 consists of Eqs. (24a) – (24k) with the roots of Eq. (19) and \( c_0 = a_1 p_0 / (p_0 + a_2) \) for DGD.

Solution 2 is the same as solution 1 but has \( c_0 = a p_0 \) for IGD.

Solution 3 equals solution 1 with Eqs. (25a) – (25d) and \( \beta_n = 0, \pi, 2\pi, ..., n\pi \).

Solution 4 is the component \( h_{SHM} \) of solution 1 for DGD.

Solution 5 consists of Eqs. (40a) – (40c) for IGD.

Solution 6 consists of Eqs. (40a) – (40c) with \( H(f,\bar{z}) \) defined by Eq. (45).

<sup>a</sup> \( \bar{z}_u = 1 \) and \( \bar{z}_l = 0 \) for fully screened well.

<sup>b</sup> The solution is independent of elevation due to fully screened well.
Table 2. Hydraulic parameters estimated by the present solution for OPT data from the Savannah River site

<table>
<thead>
<tr>
<th>Observation well</th>
<th>Present solution</th>
<th>$T$ (m$^2$/s)</th>
<th>$S$</th>
<th>$K_v$ (m/s)</th>
<th>$S_y$</th>
<th>$\varepsilon$ (s$^{-1}$)</th>
<th>SEE</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Surficial Aquifer</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>101D Solution 3$^a$</td>
<td>9.27 x 10$^{-4}$</td>
<td>2.44 x 10$^{-3}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.018</td>
<td>-5.56 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>Solution 6$^a$</td>
<td>9.18 x 10$^{-4}$</td>
<td>2.33 x 10$^{-3}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.018</td>
<td>-2.20 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>Solution 4$^b$</td>
<td>4.61 x 10$^{-4}$</td>
<td>3.95 x 10$^{-3}$</td>
<td>7.38 x 10$^{-6}$</td>
<td>2.23 x 10$^{-3}$</td>
<td>1.06 x 10$^{-2}$</td>
<td>0.018</td>
<td>-2.20 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>Solution 5$^c$</td>
<td>5.25 x 10$^{-4}$</td>
<td>1.09 x 10$^{-3}$</td>
<td>2.61 x 10$^{-5}$</td>
<td>5.49 x 10$^{-3}$</td>
<td>-</td>
<td>0.019</td>
<td>-2.30 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>102D Solution 3$^a$</td>
<td>9.13 x 10$^{-4}$</td>
<td>1.76 x 10$^{-3}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.010</td>
<td>-4.38 x 10$^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Solution 6$^a$</td>
<td>9.17 x 10$^{-4}$</td>
<td>1.67 x 10$^{-3}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.011</td>
<td>9.57 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>Solution 4$^b$</td>
<td>9.57 x 10$^{-5}$</td>
<td>7.85 x 10$^{-4}$</td>
<td>3.68 x 10$^{-6}$</td>
<td>4.95 x 10$^{-3}$</td>
<td>2.38 x 10$^{-3}$</td>
<td>0.011</td>
<td>9.57 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td>Solution 5$^c$</td>
<td>9.49 x 10$^{-5}$</td>
<td>3.25 x 10$^{-4}$</td>
<td>4.67 x 10$^{-6}$</td>
<td>4.68 x 10$^{-3}$</td>
<td>-</td>
<td>0.011</td>
<td>9.50 x 10$^{-4}$</td>
<td></td>
</tr>
<tr>
<td><strong>Barnwell-McBean Aquifer</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>201C Solution 3$^a$</td>
<td>5.86 x 10$^{-5}$</td>
<td>7.07 x 10$^{-4}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.232</td>
<td>0.046</td>
<td></td>
</tr>
<tr>
<td>Solution 6$^b$</td>
<td>6.03 x 10$^{-5}$</td>
<td>6.54 x 10$^{-4}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.363</td>
<td>0.281</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ transient confined flow
$^b$ PSS confined flow
$^c$ PSS unconfined flow
Figure 1. Schematic diagram for oscillatory pumping test at a partially screened well of finite radius in an unconfined aquifer.
Figure 2. Influence of delayed gravity drainage on the dimensionless amplitude $A_\bar{t}$ and transient head $\bar{h}$ at $\bar{r} = 1$, $\bar{z} = 1$ predicted by solution 1 for different magnitudes of $a_1$ related to the influence.
Figure 3. Relative error (RE) on the dimensionless amplitudes \( \tilde{A}_t \) at the rim of the pumping well (i.e., \( r = r_w \)) predicted by the Dagan and Rabinovich (2014) solution and the IGD solution. The well radius is assumed infinitesimal in the Dagan and Rabinovich (2014) solution and finite in our solution.
Figure 4. The normalized sensitivity coefficient $S_i$ associated with (a) the exponential component $h_{exp}$ of solution 1 and (b) the SHM amplitude $A_t$ for parameters $K_r$, $K_z$, $S_x$, $S_y$, $\omega$ and $\varepsilon$. The observation locations for panels (a) and (b) are under water table (i.e., $\bar{z} = 0.5$). Panel (c) displays the curves of $S_i$ of $h_{exp}$ and $A_t$ at water table (i.e., $\bar{z} = 1$) versus $S_y$ and $\varepsilon$. 
Figure 5. Heads fluctuations at $\bar{r} = 6$ predicted by (a) DGD solution 1 and (b) IGD solution 2. Solutions 1 and 2 are expressed as $\bar{h} = \bar{h}_{exp} + \bar{h}_{SHM}$ for transient flow. IGD solution 5 expressed as $\bar{h}_s = A_s \cos(\gamma t - \phi_s)$ accounts for PSS flow.
Figure 6. Heads fluctuations at $\bar{r} = 6$ predicted by solutions 3 and 6 for (a) partially-screend pumping well and (b) fully-screened pumping well. Solution 3 is expressed as $\bar{h} = \bar{h}_{\text{exp}} + \bar{h}_{\text{SHM}}$ for transient flow. Solution 6 expressed as $\bar{h}_s = A_s \cos(\gamma t - \phi_s)$ accounts for PSS flow.
Figure 7. Comparison of field observation data with head fluctuations predicted by the present solution. Solutions 3 and 6 represent transient and PSS confined flows, respectively. PSS solutions 4 and 5 stand for DGD and IGD conditions, respectively.
Figure A1. Modified Bromwich contour for the inverse Laplace transform to a multiple-value function with a branch point and a branch cut.